A CHARACTERIZATION OF THE CONTACT LIE ALGEBRAS

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ABSTRACT. We classify the simple finite-dimensional irreducible graded Lie algebras over an algebraically closed field of characteristic $p > 5$ which have the form $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k$, $k \geq 3$, where L_0 is classical and reductive. We show that any such Lie algebra must be a Lie algebra of the contact series of Lie algebras of Cartan type by showing how the constraints imposed by the hypotheses force the existence of a highest-weight vector in L_{-1} for the representation of L_0 in L_{-1} induced by the adjoint representation of L in itself. The existence of this highest-weight vector enables us to conclude that the above-mentioned representation is restricted. L can then be determined by appeal to an earlier classification theorem.

In [1], the present author classified the simple finite-dimensional graded Lie algebras over an algebraically closed field of characteristic $p > 3$ which have the form $L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k$ where L_0 is classical and reductive. In this paper we classify the simple finite-dimensional irreducible graded Lie algebras over an algebraically closed field of characteristic $p > 5$ which have the form

$$L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k, \quad k \geq 3; \quad (1)$$

here, too, we assume that L_0 is classical and reductive. We in fact prove the following

THEOREM. *Let L be a simple finite-dimensional irreducible graded Lie algebra of the form (1) over an algebraically closed field of characteristic $p > 5$, and suppose that L_0 is classical and reductive; that is, that L_0 is the direct sum of simple, classical Lie subalgebras and a center of dimension at most one. Then L is a Lie algebra of the contact series of Lie algebras of Cartan type.*

REMARK. Let ρ be the representation of L_0 in L_{-1} which is induced by the adjoint representation of L in itself. When we say that L is irreducible, we mean that ρ is irreducible; that is, L_{-1} is a nonzero L_0 -module which contains no proper L_0 -submodules. (When $L_{-1} = \{0\}$, it is easy to see—e.g., from the proof of Lemma 1 below—that $L_i = \{0\}$ for all odd i , so that L has the form of the Lie algebras classified in [1].) When ρ is a p -representation, the above theorem is a consequence of Theorem 3 of [2]. This paper is devoted to showing, by a series of lemmas, that the assumption of restrictedness need not be made in this case.

We fix a system of simple (positive) roots $\{\alpha_i\}_{i=1}^n$ of L_0 , and we denote by B^+ and B^- the Borel subalgebras associated with the sets of positive and negative

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roots, respectively, which are determined by this choice of simple roots. We select root vectors e_i, f_i , and h_i , associated with the roots $\alpha_i, -\alpha_i$, and 0, respectively, such that $[e_i, f_i] = \delta_{ij}h_i$, $[e_i, h_i] = 2e_i$, and $[f_i, h_i] = -2f_i$, $1 \leq i \leq n$.

LEMMA 1. *If L is a simple, graded Lie algebra of the form (1), then L_{-2} is an irreducible L_0 -module. Furthermore, $[L_{-1}, L_{-1}] \neq \{0\}$, so $[L_{-1}, L_{-1}] = L_{-2}$. In addition, we have that $L_{-1} = [L_{-2}, L_1]$. Lastly, $[L_{-1}, x] \neq \{0\}$ for all $x \notin L_{-2}$.*

PROOF. If $[x, L_i] = \{0\}$ for all $i < 0$, then

$$I_x = \langle x \rangle + \sum_{\substack{i > 0 \\ j_1, \dots, j_i > 0}} [\dots [x, L_{j_1}], \dots, L_{j_i}]$$

is an ideal of L , so that I_x equals L . Hence, if $x \in L_j$ for some $j > -1$, and $[x, L_i] = \{0\}$ for all $i < 0$, then we would have $L = I_x \subseteq L_j \oplus L_{j+1} \oplus \dots \oplus L_k$, a contradiction. It follows that for all $j > -1$, we have that for all nonzero $x \in L_j$, the subspace $[L_i, x]$ is nonzero for some $i < 0$. In particular, since $[L_{-2}, L_{-1}] = \{0\}$, we must have $[L_{-1}, L_{-1}] \neq \{0\}$.

On the other hand, for any $x \in L_{-2}$, we have $I_x = L$, so that $L_{-2} \subseteq I_x$ (which implies that L_{-2} is irreducible) and $L_{-1} \subseteq I_x$ (which implies that $L_{-1} = [L_{-2}, L_1]$). Since $[L_{-1}, L_{-1}]$ is a nonzero submodule of L_{-2} , we must have $L_{-2} = [L_{-1}, L_{-1}]$. Then by the Jacobi identity, we have that $[L_{-1}, x] = \{0\}$ implies $[L_{-2}, x] = \{0\}$. Consequently, for any $j > -1$ and any $x \in L_j$, we have from the preceding paragraph that $[L_{-1}, x] \neq \{0\}$. Q.E.D.

LEMMA 2. *If L is a simple, graded Lie algebra of the form (1), then L_k is an irreducible L_0 -module, and $L_{j-1} = [L_j, L_{-1}]$ for all j , $-1 \leq j \leq k$. If L is also irreducible, and the characteristic of the base field is greater than 3, then L is a transitive Lie algebra; that is,*

- (a) $[L_{-1}, w] \neq \{0\}$ for all $w \in L_{j'} \setminus \{0\}$, $0 \leq j' \leq k$, and
- (b) $[L_1, z] \neq \{0\}$ for all $z \in L_{j''} \setminus \{0\}$, $-2 \leq j'' \leq 0$.

PROOF. If $[y, L_i] = \{0\}$ for all $i > 0$, then

$$J_y = \langle y \rangle + \sum_{\substack{i > 0 \\ j_1, \dots, j_i < 0}} [\dots [y, L_{j_1}], \dots, L_{j_i}]$$

is an ideal of L , so that J_y equals L . Since $L_{-2} = [L_{-1}, L_{-1}]$ by Lemma 1, we have that

$$J_y = \langle y \rangle + \sum_{\substack{m > 0 \\ r > 0}} \left[\dots \left[\underbrace{[\dots [y, L_0], \dots, L_0]}_{m \text{ copies}}, \underbrace{L_{-1}, \dots, L_{-1}}_{r \text{ copies}} \right] \right],$$

so that $L_{j-1} = [L_j, L_{-1}]$.

It remains to show that L is transitive. Lemma 1 shows that (a) holds. That (b) is true is established by the following proof.

I. $\text{Ann}_{L_j} L_1$ is an L_0 -submodule of L_j , so by the irreducibility of L_{-1} (by hypothesis) and L_{-2} (by Lemma 1), $\text{Ann}_{L_j} L_1 = \{0\}$ or L_j for $j'' < 0$. By Lemma 1, $\text{Ann}_{L_j} L_1 = \{0\}$ for $j'' < 0$.

II. Let $B = \text{Ann}_{L_0} L_1$. If $B \neq \{0\}$, then $[L_{-1}, B]$ is a nonzero L_0 -submodule of L_{-1} , so $[L_{-1}, B] = L_{-1}$. Then $[L_{-1}, L_1] = [[L_{-1}, B], L_1] \subseteq [[L_{-1}, L_1], B] + [L_{-1}, [B, L_1]] \subseteq [L_0, B] \subseteq B$. Then by the first part of this lemma, $B = L_0$. Thus, if $y \in L_1$, then $[L_{-1}, y]$ is an ideal of L_0 . But if $x \in L_{-1}$, then $\{0\} = (\text{ad } y)^3 (\text{ad } x)^3 L_0 = 6(\text{ad}[y, x])^3 L_0$. Thus, $(\text{ad}[y, x])^3 = 0$. Thus, $[L_{-1}, y]$ is a nil ideal, and as L_0 acts faithfully and irreducibly on L_{-1} , Engel's Theorem implies that $[L_{-1}, y] = \{0\}$. Hence, $[L_{-1}, L_1] = \{0\}$, which contradicts (a). Q.E.D.

LEMMA 3. We have $[L_{-2}, L_i] \neq \{0\}$ for all i , $0 < i < k$, and $[L_0, L_k] \neq \{0\}$.

PROOF. Because for $y \in L_k$, J_y above equals L , we have from the fact that $[L_{-2}, L] \neq \{0\}$ that $[L_{-2}, L_k] \neq \{0\}$. Since

$$\underbrace{\left[L_{-1}, \dots, [L_{-1}, [L_{-2}, L_k]] \right]}_{k \text{ copies}} \cdots$$

is nonzero and equal to

$$\underbrace{\left[L_{-1}, \dots, [L_{-1}, [L_{-2}, [L_{-1}, \dots, [L_{-1}, L_k] \cdots]]] \right]}_{i \text{ copies}} \cdots \underbrace{\left[L_{-1}, \dots, [L_{-1}, L_k] \cdots \right]}_{k-i \text{ copies}}$$

for all i , $0 < i < k$, we have the first assertion of the lemma. The proof that $[L_k, L_0] \neq \{0\}$ is similar, using the fact that (by the first part of this lemma) $[L_{-2}, L_k] \neq \{0\}$, together with the fact that for any $y \notin L_k$, there exists an $i > 0$ such that $[L_i, y] \neq \{0\}$ (see the construction of J_y above). Q.E.D.

LEMMA 4. For all i , $-2 < i < k-1$, L_i is spanned by weight vectors of the form $[v, x]$, where v is a weight vector for the representation of L_0 in L_{-1} and x is a weight vector for the representation of L_0 in L_{i+1} . L_k is spanned by weight vectors also.

PROOF. In view of Lemma 2 and our assumption that L_{-1} is irreducible as an L_0 -module, this follows as in the proof of Lemma 7 of [1]. Q.E.D.

We denote the annihilator of a submodule S in a submodule T by $\text{Ann}_T S$. Then we have the following

LEMMA 5. Let $i > 0$, and suppose that $\text{Ann}_{L_i} L_{-2} \neq \{0\}$. Then $\text{Ann}_{L_{i-1}} L_{-2} \neq \{0\}$.

PROOF. By Lemma 1, $[L_{-1}, \text{Ann}_{L_i} L_{-2}] \neq \{0\}$. But

$$[L_{-2}, [L_{-1}, \text{Ann}_{L_i} L_{-2}]] = [L_{-1}, [L_{-2}, \text{Ann}_{L_i} L_{-2}]] = \{0\},$$

so that $(\{0\} \neq) [L_{-1}, \text{Ann}_{L_i} L_{-2}] \subseteq \text{Ann}_{L_{i-1}} L_{-2}$. Q.E.D.

LEMMA 6. If $\text{Ann}_{L_1} L_{-2} \neq \{0\}$, then L_{-1} contains a highest-weight vector v^+ for the representation of L_0 in L_{-1} ; i.e., $[B^+, v^+] \subseteq \langle v^+ \rangle$.

PROOF. Since $\text{Ann}_{L_1} L_{-2}$ is an L_0 -submodule of L_1 , $M = [L_{-1}, \text{Ann}_{L_1} L_{-2}]$ is an L_0 -submodule of L_0 . M is nonzero by Lemma 1, so M is the direct sum of certain of the direct summands of L_0 (including, possibly, the center). Let r^+ be a root vector of M such that $[B^+, r^+] \subseteq \langle r^+ \rangle$ (r^+ could be a highest-root vector of one of the simple direct summands of M , if such there be, or r^+ could be an element of the center of L_0 , if the latter is contained in M). Then, since the nonzero-root spaces of L_0 are one-dimensional (and the center of L_0 is assumed to be at most one-dimensional), we have, in view of the definition of M and Lemma 4, that $r^+ = [v, a]$, where v and a are weight vectors of L_{-1} and $\text{Ann}_{L_1} L_{-2}$, respectively. Since $[L_{-1}, r^+] \neq \{0\}$ by Lemma 1, there exists a weight vector $v_0 \in L_{-1}$ such that $[v_0, r^+] \neq 0$ by Lemma 4. Because the operators $\text{ad } L_{-1}$ commute on $\text{Ann}_{L_1} L_{-2}$, the existence of a highest-weight vector for the representation of L_0 in L_{-1} follows as in the proof of Lemma 8 of [1]. Q.E.D.

LEMMA 7. If $\text{Ann}_{L_j} L_{-2} = \{0\}$ for all $j \geq 0$, then L_{-1} contains a highest-weight vector v^+ for the representation of L_0 in L_{-1} .

PROOF. Since $\text{Ann}_{L_0} L_{-2} = \{0\}$, an argument similar to that used in the proof of Lemma 8 of [1] shows that L_{-2} contains a highest-weight vector for the representation of L_0 in L_{-2} (begin the argument by choosing $r^+ \in [L_{-2}, L_2]$, which is nonzero by Lemma 3). Then an argument similar to that used in §4 of [1] shows that the representation of L_0 in L_{-2} is restricted. Thus, in view of Lemma 1 and Kac's remark in §5 of [2],

$$E = L_{-2} \oplus L_0 \oplus L_2 \oplus \cdots \oplus L_{2[k/2]}$$

is an irreducible, transitive, graded Lie algebra such that the representation of L_0 in L_{-2} is restricted. Then Theorem 3 of [2] shows that E is classical or of Cartan type.

If $k > 3$, then $2[k/2] \geq 4$, so E must be of Cartan type, and L_0 is the direct sum of its (possibly trivial) center and a simple Lie algebra S of type A_n or C_n ; furthermore, the representation of S on L_{-2} is the natural representation of S .

Using Lemma 2, we can write, as before, $r^+ = [v, c]$, where r^+ is a highest-root vector of S , and where v and c are weight vectors for the representations of L_0 in L_{-1} and L_1 , respectively. Since $[L_{-1}, r^+] \neq \{0\}$, we know that there exists a weight vector v_0 in L_{-1} such that $[v_0, r^+] \neq 0$. Suppose there were an infinite sequence of vectors $e_{i_1}, e_{i_2}, \dots$, associated with the simple (positive) roots $\alpha_{i_1}, \alpha_{i_2}, \dots$, respectively, such that

$$v'_j = [\cdots [[v_0, r^+], e_{i_1}], e_{i_2}], \dots, e_{i_j}] \neq 0$$

for all $j \geq 1$. If we define $v_j = [\cdots [v_0, e_{i_1}], \dots, e_{i_j}]$, we have $(0 \neq) v'_j = [v_j, r^+] = [[v_j, v], c] + [v, [v_j, c]]$ for all $j \geq 1$; i.e., for all $j \geq 1$, either $[v_j, v] \neq 0$ or $[v_j, c] \neq 0$. Let w_0 be the sum of the weights of v_0 and v , and let β_0 be the sum of the weights of v_0 and c . For $j \geq 1$, we recursively define $w_j = w_{j-1} + \alpha_{i_j}$ and $\beta_j = \beta_{j-1} + \alpha_{i_j}$. Then when $[v_j, v]$ is nonzero, it has weight w_j ; similarly, when $[v_j, c]$ is nonzero, it has root β_j .

Now suppose that the weight vectors $\{[v_{j+i}, v]\}_{i=0}^t$ are all nonzero, and let $w_j, w_{j+1}, \dots, w_{j+t}$ be their corresponding weights. Let $w_A^-, w_A^+, w_C^-,$ and w_C^+ be the lowest and highest weights of A_n and C_n , respectively, and set

$$\begin{aligned} w_1^{(1)} &= w_A^-, & w_2^{(1)} &= w_A^- + \alpha_n, \\ w_3^{(1)} &= w_A^- + \alpha_n + \alpha_{n-1}, \dots, \\ w_{n+1}^{(1)} &= w_A^- + \alpha_n + \alpha_{n-1} + \dots + \alpha_1 = w_A^+; \end{aligned} \quad (2)$$

$$\begin{aligned} w_1^{(2)} &= w_A^-, & w_2^{(2)} &= w_A^- + \alpha_1, \\ w_3^{(2)} &= w_A^- + \alpha_1 + \alpha_2, \dots, \\ w_{n+1}^{(2)} &= w_A^- + \alpha_1 + \alpha_2 + \dots + \alpha_n = w_A^+; \end{aligned} \quad (3)$$

$$\begin{aligned} w_1^{(3)} &= w_C^-, & w_2^{(3)} &= w_C^- + \alpha_1, \\ w_3^{(3)} &= w_C^- + \alpha_1 + \alpha_2, \dots, \\ w_{2n}^{(3)} &= w_C^- + \alpha_1 + \alpha_2 + \dots + \alpha_{n-1} + \alpha_n + \alpha_{n-1} + \dots + \alpha_1 = w_C^+. \end{aligned} \quad (4)$$

Since $w_b^{(d)} \neq w_X^+ + \alpha_l$, $X = A$ or C , for all integers b, d , and l for which the expression is defined, it follows that if $S = A_n$, we must have an integer m such that $w_{j+i} = w_{m+i}^{(1)}$, $i = 0, \dots, t$, or $w_{j+i} = w_{m+i}^{(2)}$, $i = 0, \dots, t$; similarly, if $S = C_n$, we must have an m such that $w_{j+i} = w_{m+i}^{(3)}$, $i = 0, \dots, t$. In particular, it must eventually occur that, for some integer s , w_s is a weight, w_{s+1} is not a weight, and if q is the smallest positive integer such that w_{s+q} is a weight, then $w_s = w_b^{(d)}$ and $w_{s+q} = w_{b'}^{(d)}$, where $b' \leq b$ ($d = 1, 2$, or 3). Then $\beta_{s+1}, \beta_{s+2}, \dots, \beta_{s+q-1}$ must be a root string of L_0 such that $w_{s+q} - w_s = \alpha_{i+1} + (\beta_{s+q-1} - \beta_{s+1}) + \alpha_{i+q}$. In view of (2), (3), and (4), $-\alpha_i$ can appear no more than twice as a summand of $w_{s+q} - w_s$ for any i , $1 \leq i \leq n$. Thus, $\alpha_{i+1} + (\beta_{s+q-1} - \beta_{s+1}) + \alpha_{i+q} = \sum_{i=1}^n t_i \alpha_i$, where $t_i \equiv p - 2$, $t_i \equiv p - 1$, or $t_i \equiv 0 \pmod{p}$. Then $\beta_{s+q-1} - \beta_{s+1}$ must contain at least $p - 4 \geq 3$ copies of each simple root appearing in it. Inasmuch as $\beta_{s+q-1} - \beta_{s+1}$ is the sum of the roots in at most two contiguous subarrays of the following array:

$$\begin{array}{ccccccc} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & & & \\ & & & & \alpha_n & & \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & & & \end{array}$$

it is clear that we have arrived at a contradiction. (Note that if any contiguous subarray contains more than two copies of the same simple root, then it must be a subarray of the second array and contain at most two copies of α_n .) Thus, no such infinite sequence as $\{[v_j, r^+]\}_{j=1}^\infty$ can occur, and we must arrive at a vector $v^+ = [v_u, r^+]$, $u \geq 1$, which is annihilated by $\{e_i\}_{i=1}^n$.

Now suppose that $k = 3$. By the above-quoted Theorem 3 of [2], there exists an element d^+ of L_2 which is a highest-root vector of the classical Lie algebra E , and, hence, a highest-weight vector for the representation of L_0 in L_2 . By Lemma 2 (and the fact that the nonzero-root spaces of a classical Lie algebra are one-dimensional), it follows that $d^+ = [v, e]$, where v and e are weight vectors of L_{-1} and L_3 , respectively. Letting $y = d^+$ in the proof of Lemma 2, we get that $[d^+, L_1] \neq \{0\}$. Let c be a weight vector of L_1 such that $e' = [d^+, c] \neq 0$, and

suppose there existed an infinite sequence of simple (positive) root vectors $e_{i_1}, e_{i_2}, \dots$, such that $[\dots [e', e_{i_1}], \dots, e_{i_j}] \neq 0$ for all $j \geq 1$. Then

$$\begin{aligned} (0 \neq) \quad [\dots [e', e_{i_1}], \dots, e_{i_j}] &= [\dots [[d^+, c], e_{i_1}], \dots, e_{i_j}] \\ &= [d^+, [\dots [c, e_{i_1}], \dots, e_{i_j}]] \\ &= [v, e], [\dots [c, e_{i_1}], \dots, e_{i_j}] \\ &= [v, [\dots [c, e_{i_1}], \dots, e_{i_j}], e] \end{aligned}$$

for all $j \geq 1$. But then $\{[v, [\dots [c, e_{i_1}], \dots, e_{i_j}]]\}_{j=1}^\infty$ is an infinite sequence of root vectors of L_0 , each of whose roots differs from the previous root by a simple positive root. Since such a sequence cannot exist in the classical L_0 , we see that L_3 must contain a highest-weight vector e^+ . As we noted at the beginning of the proof of this lemma, L_{-2} contains a highest-weight vector w^+ . Since we have assumed that $\text{Ann}_{L_3} L_{-2} = \{0\}$, we have that $[L_{-2}, e^+] \neq \{0\}$. By Lemma 1, L_{-2} is irreducible as an L_0 -module, so it is spanned by vectors of the form $[\dots [w^+, f_{i_1}], \dots, f_{i_j}]$. Let j be minimal such that $c^+ = [[\dots [w^+, f_{i_1}], \dots, f_{i_j}], e^+] \neq 0$. Then c^+ is a highest-weight vector of L_1 . Similarly, if j' is minimal such that $v^+ = [[\dots [w^+, f_{i'_1}], \dots, f_{i'_j}], c^+] \neq 0$, then v^+ is a highest-weight vector of L_{-1} , as required. Q.E.D.

LEMMA 8. *If $\text{Ann}_{L_i} L_{-2} = \{0\}$ for all $i \geq 1$, and $\text{Ann}_{L_0} L_{-2} \neq \{0\}$, then L_{-1} contains a highest-weight vector v^+ for the representation of L_0 in L_{-1} .*

PROOF. Set $A = \text{Ann}_{L_0} L_{-2}$. Since $\text{Ann}_{L_i} L_{-2} = \{0\}$ for all $i \geq 1$, we have that $[[L_3, L_{-2}], L_{-2}] \neq \{0\}$. Since L_{-1} is assumed to be irreducible, we have that $L_{-1} = [[L_3, L_{-2}], L_{-2}]$. By Lemma 2, we have that $L_0 = [L_{-1}, L_1]$, so

$$\begin{aligned} L_0 &= [L_{-1}, L_1] = [[[L_3, L_{-2}], L_{-2}], L_1] \\ &\subseteq [[[L_3, L_1], L_{-2}], L_{-2}] + [[[L_3, [L_{-2}, L_1]], L_{-2}] + [[[L_3, L_{-2}], [L_{-2}, L_1]] \\ &\subseteq [[[L_4, L_{-2}], L_{-2}] + [[[L_3, L_{-1}], L_{-2}] + [[[L_3, L_{-2}], L_{-1}]] \\ &= [[[L_4, L_{-2}], L_{-2}] + [[[L_3, L_{-1}], L_{-2}] + [[[L_3, L_{-1}], L_{-2}]] \\ &\subseteq [L_2, L_{-2}]. \end{aligned}$$

Hence, $L_0 = [L_2, L_{-2}]$.

Now consider $D = [A, L_2]$. We have

$$[L_{-2}, D] = [L_{-2}, [A, L_2]] = [A, [L_{-2}, L_2]] = [A, L_0] = A \neq \{0\} \quad (5)$$

(A cannot contain the center, since each nonzero element of the center acts as a nonzero scalar on L_{-2}). Let r^+ be a highest-root vector of A . Then, as before, $r^+ = [w, d]$, where w is a weight vector of L_{-2} and d is a weight vector of D . Furthermore, since every element of L_{-2} has A -weight zero, r^+ and d have the same A -weight. Since $L_{-2} = [L_{-1}, L_{-1}]$ by Lemma 1, it follows that w is a linear combination of weight vectors of the form $[v, v']$, where v and v' are weight vectors of L_{-1} . Since the nonzero-root spaces of L_0 are one-dimensional, we must have that

$r^+ = [[v, v'], d]$ for certain v and v' . By the Jacobi identity, $r^+ = [v, [v', d]] - [v', [v, d]]$, so that by relabelling and/or multiplying one of the vectors v or v' by a scalar, we can assume that $r^+ = [v, [v', d]]$. Since r^+ and d have the same A -weight, the A -weight of v must be the negative of the A -weight of v' . Now, by Lemma 1, $[L_{-1}, r^+] \neq \{0\}$, so there exists a $v_0 \in L_{-1}$ such that $[v_0, r^+] = [v_0, [v, [v', d]]] \neq 0$. Suppose there were an infinite sequence $e_{i_1}, e_{i_2}, \dots$ of simple (positive) root vectors such that

$$v'_j = [\dots [[v_0, r^+], e_{i_1}], \dots, e_{i_j}] \neq 0 \quad \text{for all } j \geq 1.$$

If we define $v_j = [\dots [v_0, e_{i_1}], \dots, e_{i_j}]$, then we have

$$(0 \neq) \quad v'_j = [v_j, r^+] = [[v_j, v], [v', d]] + [v, [v_j, [v', d]]].$$

Set $s_j = [[v_j, v], [v', d]]$ and $t_j = [v, [v_j, [v', d]]]$. Then for all $j \geq 1$, either $s_j \neq 0$ or $t_j \neq 0$.

If $s_j \neq 0$, then $[v_j, v] \neq 0$. But $[v_j, v] \in L_{-2}$, so $[v_j, v]$ has zero A -weight. Thus, the A -weight of v_j must be the negative of that of v ; i.e., the A -weight of v_j must equal the A -weight of v' . Now, $(0 \neq) s_j = [[v_j, v], [v', d]] = [v', [[v_j, v], d]]$. Furthermore, by (5), $[[v_j, v], d] \in A$, and since $[[v_j, v], d]$ has the same A -weight as $[v, [v', d]] = r^+$, we must have $(0 \neq) [[v_j, v], d] \in \langle r^+ \rangle$.

Thus, v_j has the same L_0 -weight as v' ; that is, the only occasion on which $s_j \neq 0$ is when v_j has the same L_0 -weight as v' .

If $t_j \neq 0$, then $r_j = [v_j, [v', d]]$ is a nonzero-root vector of L_0 . But by the preceding two paragraphs, $t_j = 0$ only when v_j has L_0 -weight equal to v' . Thus, if j' is the first subscript for which $t_{j'} = 0$ and j'' is the second, then the L_0 -weight of $v_{j'}$ is equal to the L_0 -weight of $v_{j''}$, so that we have a sequence of root vectors $r_{j'+1}, r_{j'+2}, \dots, r_{j''-1}$ of L_0 such that the root associated with $r_{j'+1}$ differs (modulo p) from the root associated with $r_{j''-1}$ by the sum of two simple roots ($\alpha_{j'+1}$ and $\alpha_{j''}$). Thus, each simple (positive) root in the sequence $\alpha_{j'+2}, \dots, \alpha_{j''-1}$ appears at least $p - 2 \geq 5$ times. Since no such root string can occur in a classical Lie algebra, no such sequence as $\{v'_j\}_{j=1}^\infty$ can occur; i.e., any such sequence must terminate in a vector v^+ which is annihilated by all of $\{e_i\}_{i=1}^n$. Then v^+ is the required highest-weight vector. Q.E.D.

PROOF OF THEOREM. By Lemmas 5, 6, 7, and 8, we have that L_{-1} contains a highest-weight vector for the representation of L_0 in L_{-1} . Then the theorem follows as in §4 of [1]. Q.E.D.

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