BOOLEAN POWERS OF GROUPS

JOHN LAWRENCE¹

ABSTRACT. A group is *B*-separating if a Boolean power of the group has a unique Boolean algebra. It is proved that a finite subdirectly irreducible group is *B*-separating if and only if it is non-Abelian.

Suppose *B* is a Boolean ring and *G* is a group. Let *B*[*G*] denote the group ring of *G* with coefficient ring *B*. The Boolean power *G*[*B*] is defined to be the set of those elements

$$\sum e_i g_i \in B[G]$$

such that

- (1) $\sum e_i = 1$,
- (2) $e_i e_j = 0$ if $g_i \neq g_j$.

The support of the element is the set $\{g_i \in G | e_i \neq 0\}$. *G*[*B*] is a group considered as a subgroup of the group of units of *B*[*G*]. Consult the survey article [1] for basic properties of Boolean powers.

A group *G* is said to be *B*-separating if $G[B] \cong G[B']$ implies $B \cong B'$ for all Boolean rings *B* and *B'* [1].

In this note we prove:

- (1) No Abelian group is *B*-separating;
- (2) A finite subdirectly irreducible group is *B*-separating if and only if it is non-Abelian. On the other hand, given a group *G*, the group $G \times G$ is not *B*-separating.

It is known that no finite Abelian group is *B*-separating. Neumann and Yamamuro have proved that a countable non-Abelian simple group is *B*-separating [6], while Jonsson has proved that a countable centerless indecomposable group is *B*-separating [3].

After this paper had been submitted, the author discovered that G. Bergman [7] had proved that no Abelian group is *B*-separating. Because of its simplicity, we have retained our original proof of this fact.

If *B* is a Boolean ring, then there is a partial order $\leq$ on *B* defined by $e \leq f$ if $ef = e$. Now consider *B* as a vector space over Z_2 , the two-element field. We say that *B* has a *totally ordered basis* if *B* has a totally ordered basis (under $\leq$) as a vector space over Z_2 . It is well known that every countable Boolean ring has a totally ordered basis [5].

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LEMMA 1. Let B be a Boolean ring with a totally ordered basis $T = \{t_i\}_{i \in \mathcal{C}}$ and let G be an Abelian group. Then an element x of $G[B]$ has a unique representation of the form

$$x = [(1 - t_1) + t_1 g_1][(1 - t_2) + t_2 g_2] \cdots [(1 - t_k) + t_k g_k], \quad (*)$$

where $t_1 > t_2 > \cdots > t_k$, $g_i \neq 1$, and $k > 0$.

PROOF. Clearly $\sum_{i=1}^n e_i g_i = \prod_{i=1}^n [(1 - e_i) + e_i g_i]$, so we need to show that every element of the form $(1 - e) + eg$ has a representation of the form (*). Suppose $e = t_1 + t_2 + \cdots + t_k$, where $t_1 > t_2 > \cdots > t_k$. Then

$$\begin{aligned} [(1 - e) + eg] &= [(1 - t_1) + t_1 g][(1 - t_2) + t_2 g^{-1}][(1 - t_3) + t_3 g] \\ &\quad \cdots [(1 - t_k) + t_k g^{(-1)^{k+1}}]. \end{aligned}$$

We have proved that every element of $G[B]$ has a representation of the form (*).

To show that this representation is unique, we must show that if $k > 0$, then $x \neq 1$. But $t_1(1 - t_2) \cdots (1 - t_k) = t_1(1 - t_2) \neq 0$, so if $k > 0$, g_1 occurs in the support, hence $x \neq 1$.

THEOREM 2 (BERGMAN [7]). No Abelian group is B -separating.

PROOF. Let B and B' be two nonisomorphic Boolean rings with the same cardinality and totally ordered bases, $\{t_i\}_{i \in \mathcal{C}}$ and $\{s_i\}_{i \in \mathcal{C}}$ respectively. Let G be an Abelian group. Let α be a bijection from the set $\{t_i\}_{i \in \mathcal{C}}$ to the set $\{s_i\}_{i \in \mathcal{C}}$. Define

$$\phi[(1 - t_i) + t_i g] = [(1 - \alpha(t_i)) + \alpha(t_i)g]$$

and extend this multiplicatively. It is easily proved, using Lemma 1, that ϕ extends to an isomorphism from $G[B]$ to $G[B']$. Since $B \not\cong B'$, while $G[B] \cong G[B']$, we have shown that G is not B -separating.

Theorem 2 can be generalized to show that no affine algebra [4] is B -separating.

Let G be a non-Abelian finite subdirectly irreducible p -group with $|G| = p^n$. Let C be the center of G , K be the minimal nontrivial normal subgroup of G and A be the second element in the upper central series of G . Let B be a Boolean ring. For each $x \in A[B]$, let

$$G_x = \{x^{(1)}x^{(2)} \cdots x^{(p^s)} | x^{(i)} \text{ a conjugate } x \text{ in } G[B], s \geq 1\}.$$

Let $K_x = K[B] \cap G_x$.

LEMMA 3. Suppose $x = \sum x_g g \in A[B]$. Then we have

$$K_x = \left\{ \sum_{g \in K} y_g g \mid \sum_{\substack{g \neq 1 \\ g \in K}} y_g < \sum_{\substack{g \in A \\ g \notin C}} x_g \right\}.$$

Hence, $K_x \supset K_x$ if and only if

$$\sum \{x_g | g \in A, g \notin C\} > \sum \{x'_g | g \in A, g \notin C\}.$$

PROOF. We first prove that $K_x \supset \{\sum_{g \in K} y_g g | \sum_{g \neq 1, g \in K} y_g < \sum_{g \in A, g \notin C} x_g\}$. Let $e = e_{gh} = y_h x_g$, $h \in K$, $g \in A \setminus C$. Let $t \in G$ be an element such that $t^{-1}gt = ag$, $1 \neq a \in C$. For some power of a , say m , $a^m = h$ (by the subdirect irreducibility of G). Then $t^{-m}gt^m = a^m g = hg$. Now,

$$x^{p^n-1}[(1-e) + et]^{-m}x[(1-e) + et]^m = (1-e) + eh \in K_x.$$

Hence,

$$\prod_{\substack{g \in A \setminus C \\ h \in K}} [(1 - e_{gh}) + e_{gh}h] = \sum_{h \in K} y_h h,$$

and this completes the first part of the proof

We now show that $K_x \subset \{\sum_{g \in K} y_g g | \sum_{g \neq 1, g \in K} y_g < \sum_{g \in A, g \notin C} x_g\}$. Let $y = \sum_{g \in K} y_g g$ be an element of K_x . Then y is a product of p^n s conjugates of x (for some s). If g is an element of C , then $g^{p^n} = 1$, thus $x_g < y_1$. Since $K \subset A$, we have that $\sum\{y_g | g \in K - \{1\}\} \leq \sum\{x_g | g \in A, g \notin C\}$.

This completes the proof of the lemma.

Now let g be a subdirectly irreducible finite group with nontrivial center C and suppose that G is not a p -group. Since G is subdirectly irreducible, the minimal nontrivial normal subgroup K is a cyclic p -group for some prime p , and its center has order p^m for some $m > 1$. Let q be a prime different from p , dividing the order of G . Let H be the set of elements of $G[B]$ of order some power of q . For each $x \in H$, let $\langle x \rangle$ be the principal normal subgroup of $G[B]$ generated by x . Finally, define

$$K_x = K[B] \cap \langle x \rangle.$$

LEMMA 4. Suppose $x = \sum_{g \in G} x_g g \in H$. Then

$$K_x = \left\{ \sum_{g \in K} y_g g \mid \sum_{\substack{g \in K \\ g \neq 1}} y_g < \sum_{\substack{g \in G \\ g \neq 1}} x_g \right\}.$$

Hence, $K_x \supset K_x$ if and only if

$$\sum\{x_g | g \in G - \{1\}\} > \sum\{x'_g | g \in G - \{1\}\}.$$

PROOF. We first prove that

$$K_x \supset \left\{ \sum_{g \in K} y_g g \mid \sum_{\substack{g \in K \\ g \neq 1}} y_g < \sum_{\substack{g \in G \\ g \neq 1}} x_g \right\}.$$

Let $e = e_{gh} = y_h x_g$, $h \in K$, $1 \neq g \in G$. Since $K \subset \langle g \rangle \subset G$ (by subdirect irreducibility), there is a product of conjugates of g equal to h , say

$$t_1^{-1}gt_1 \cdots t_l^{-1}gt_l = h, \quad t_i \in G.$$

Then $\prod_{i=1}^l [(1-e) + et_i^{-1}]x[(1-e) + et_i] = eh + \sum x''_g g$, where each g occurring in the sum has order some power of q . Since the order of h is a power of p , we have

$$(1-e) + eh \in \langle x \rangle.$$

The product

$$\prod \{ (1 - e_{gh}) + e_{gh}h | h \in K, g \in \text{supp } x, g \neq 1 \} = \Sigma \{ y_n h | h \in K \}.$$

This completes the first part of the proof.

We now show that $K_x \subset \{ \sum_{g \in K} y_g g | \sum_{g \in K, g \neq 1} y_g < \sum_{g \in G, g \neq 1} x_g \}$. The element x has ordered some power of q . Let $y = \sum y_g g \in K_x$. Then y is a product of conjugates of x . Thus $y_1 > x_1$. It follows that $\sum_{g \in K, g \neq 1} y_g < \sum_{g \in G, g \neq 1} x_g$, since $\sum_{g \in G} x_g = 1$.

This completes the proof of the lemma.

THEOREM 5. *Let G be a finite subdirectly irreducible non-Abelian group. Then G is B -separating.*

PROOF. If G is centerless, then the result follows from [3].

If G is a p -group, use Lemma 3. Since $A[B]$ is the second element in the upper central series, it can be identified in $G[B]$. By Lemma 3, the partially ordered set $\{K_x | x \in A[B]\}$ is isomorphic to B . Thus we can 'recover' B from $G[B]$, and this shows that G is B -separating.

If G is not a p -group, we use Lemma 4. Then the partially ordered set $\{K_x | x \in H\}$ is isomorphic to B , hence G is B -separating.

REMARK. It seems probable that the subdirect irreducibility in Theorem 5 can be replaced by indecomposability.

LEMMA 6. *Let B be a Boolean ring and let G and H be groups. Then $(G \times H)[B] \cong G[B] \times H[B]$.*

PROOF. Define $\phi: (G \times H)[B] \rightarrow G[B] \times H[B]$ by

$$\phi \left(\sum_i e_i (g_i \times h_i) \right) = \sum_i e_i g_i \times \sum_i e_i h_i.$$

This is an isomorphism with inverse ϕ^{-1} defined by

$$\phi^{-1} \left(\sum_i e_i g_i \times \sum_j f_j h_j \right) = \sum_{i,j} e_i f_j (g_i \times h_j).$$

PROPOSITION 7. *Let G be a group. Then the group $G \times G$ is not B -separating.*

PROOF. Let B and B' be nonisomorphic Boolean rings such that $B \times B \cong B' \times B'$ [2]. Then

$$\begin{aligned} (G \times G)[B] &\cong G[B] \times G[B] \cong G[B \times B] \cong G[B' \times B'] \\ &\cong G[B'] \times G[B'] \cong (G \times G)[B']. \end{aligned}$$

Since $B \not\cong B'$, we have shown that $G \times G$ is not B -separating.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1