PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

SOME INEQUALITIES RELATED TO HÖLDER'S INEQUALITY

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ABSTRACT. In this paper we generalize the Jodeit-Jones-Moser inequalities relating $0 \leqslant x - (\int_0^x f)^p$, if $\|f\|_q \leqslant 1$.

1. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f, \phi: X \to [0, \infty]$ be measurable. Throughout the paper $1 \leqslant q < \infty$, $1/q + 1/p = 1$, and $\|f\|_q \leqslant 1$. Also let there be given an indexing map σ: $[0, \infty) \to \mathfrak{M}$ such that $\sigma(r') \subset \sigma(r'')$, $r' \leqslant r''$, and $\sigma(0) = \varnothing$. We will write $\sigma(r) = S_r$ and we will assume that $\|\phi_r\|_p < \infty$, where $\phi_r = \phi \cdot \chi_{S_r}$.

Throughout the paper, $1 \leqslant \alpha \leqslant p$ unless $q = 1$, in which case $1 \leqslant \alpha < \infty$, and $0 \leqslant \beta < \alpha$. We set $\psi(r) = \|\phi_r\|_p$, $F(r) = \int_X f\phi_r\, d\mu$, $\psi^*(r) = \psi(r)^{\alpha-\beta}$, $F^*(r) = F(r)^\alpha\psi(r)^{-\beta}$. From Hölder's inequality, $F(r) \leqslant \psi(r)$, from which $F^*(r) \leqslant \psi^*(r)$. The main result of the paper is the following theorem.

THEOREM 1. *Let Φ: $[0, \infty) \to [0, \infty)$ be nonincreasing, and let m^* be the measure on $[0, \infty)$ induced by the nondecreasing function ψ^*. If ψ^* is continuous, then*

$$\int_0^\infty \Phi\{\psi^*(r) - F^*(r)\}\, dm^* \leqslant c\int_0^\infty \Phi(u)\, du,$$

where c depends on q, α, β only.

The special case of the above inequality for $X = [0, \infty)$, $\sigma(r) = [0, r)$, $\phi(x) \equiv 1$ on $[0, \infty)$, and $\alpha = p$, $\beta = 0$ has been studied extensively. We have then $\psi^*(r) = r$, $dm^* = dr$, and $F^*(r) = (\int_0^r f\, dx)^p$. The case $q \geqslant 2$, $\Phi(u) = e^{-u}$ is due to Moser [3], and the case $q > 1$ with $\Phi(u) = e^{-u}$ can be found in Jodeit [1]. For arbitrary nonincreasing Φ, $q > 1$, and the same special X, σ, ϕ as above, the inequality is due to Jones [2]. The proof in [2] is different from those in [1] and [3] and, as we shall see, lends itself to prove our theorem.

2. The proof of Theorem 1 is based on Theorem 2 below, a result of interest in its own right. Here the continuity of ψ^* is not required.

THEOREM 2. *There exist constants c, d depending on q, α, β only such that, for any $0 < s < \infty$, the inequalities $\psi^*(r_2) \geqslant \psi^*(r_1) \geqslant cs$, $r_2 \geqslant r_1$, and $\psi^*(r_j) - F^*(r_j) \leqslant s$, $j = 1, 2$, imply $\psi^*(r_2) - \psi^*(r_1) \leqslant ds$.*

Received by the editors February 25, 1980 and, in revised form, October 1, 1980.
1980 *Mathematics Subject Classification.* Primary 42A16; Secondary 26A14.

PROOF. We need the following inequalities.

(i) If $a_1, a_2 \geqslant 0$, $\alpha \geqslant 1$, then $a_2^\alpha - a_1^\alpha \leqslant \alpha a_2^{\alpha-1}(a_2 - a_1)$.

(ii) If $a_2 \geqslant a_1 \geqslant 0$, $\alpha_2 > \alpha_1 > 0$, $0 \leqslant \gamma < 1$, and $1 \leqslant p < \infty$, then

$$\frac{a_2}{\alpha_2^\gamma} - \frac{a_1}{\alpha_1^\gamma} \leqslant c(a_2 - a_1)\left\{ \frac{\alpha_2^{p(1-\gamma)} - \alpha_1^{p(1-\gamma)}}{\alpha_2^p - \alpha_1^p} \right\}^{1/p},$$

where $c = (1-\gamma)^{-1/p}$.

The proof of (i) is simply the mean value theorem for differentation, and for the proof of (ii) start with the expression in braces and apply the mean value theorem using the function $u^{1-\gamma}$.

We first treat the case $1 < q < \infty$. Assume first that $\psi^*(r_1) \geqslant 2s$. Then

$$\psi^*(r_1) - s \leqslant F^*(r_1) \leqslant \left(\int_{S_{r_1}} f^q \, d\mu \right)^{\alpha/q} \psi^*(r_1)$$

from which

$$\left(1 - \frac{s}{\psi^*(r_1)}\right)^{q/\alpha} \leqslant 1 - \int_{X \setminus S_{r_1}} f^q \, d\mu.$$

Thus

$$\int_{X \setminus S_{r_1}} f^q \, d\mu \leqslant 1 - \left(1 - \frac{s}{\psi^*(r_1)}\right)^{q/\alpha} \leqslant \frac{cs}{\psi^*(r_1)}, \tag{iii}$$

where c depends only on q and α. This part is, except for exponents and notation, the same as in [2].

We still assume that $\psi^*(r_1) \geqslant 2s$, and note that $0 \leqslant \psi^*(r_2) - \psi^*(r_1) \leqslant s + F^*(r_2) - F^*(r_1)$. We will estimate now

$$F^*(r_2) - F^*(r_1) = \left\{ \frac{F(r_2)}{\psi(r_2)^{\beta/\alpha}} \right\}^\alpha - \left\{ \frac{F(r_1)}{\psi(r_1)^{\beta/\alpha}} \right\}^\alpha.$$

By (i) and (ii), since we may assume that $\psi(r_2) > \psi(r_1)$, we get

$$F^*(r_2) - F^*(r_1) \leqslant c\left\{ \frac{F(r_2)}{\psi(r_2)^{\beta/\alpha}} \right\}^{\alpha-1}$$

$$\cdot [F(r_2) - F(r_1)] \cdot \left\{ \frac{\psi(r_2)^{p(1-\beta/\alpha)} - \psi(r_1)^{p(1-\beta/\alpha)}}{\psi(r_2)^p - \psi(r_1)^p} \right\}^{1/p}.$$

The numerator in braces we write as (since $\alpha \leqslant p$)

$$\psi(r_2)^{(\alpha-\beta)p/\alpha} - \psi(r_1)^{(\alpha-\beta)p/\alpha} \leqslant c\psi^*(r_2)^{p/\alpha-1}\{\psi^*(r_2) - \psi^*(r_1)\}.$$

Next,

$$F(r_2) - F(r_1) = \int_{S_{r_2} \setminus S_{r_1}} f\phi_{r_2} \leqslant \left\{ \int_{X \setminus S_{r_1}} f^q \, d\mu \right\}^{1/q} \|\phi_{r_2} \cdot \chi_{S_{r_2} \setminus S_{r_1}}\|_p.$$

Since $|\phi_{r_2} \cdot \chi_{S_{r_2} \setminus S_{r_1}}|^p = |\phi_{r_2}|^p - |\phi_{r_1}|^p$, we get from (iii) that

$$F(r_2) - F(r_1) \leqslant (cs/\psi^*(r_1))^{1/q}\{\psi(r_2)^p - \psi(r_1)^p\}^{1/p}.$$

Finally observe that $F(r_2)/\psi(r_2)^{\beta/\alpha} \leqslant \psi(r_2)^{(\alpha-\beta)/\alpha} = \psi^*(r_2)^{1/\alpha}$, and thus

$$\psi^*(r_2) - \psi^*(r_1) \leqslant s + c\psi^*(r_2)^{(\alpha-1)/\alpha} \cdot (cs/\psi^*(r_1))^{1/q} \cdot \psi^*(r_2)^{1/\alpha - 1/p} \cdot \{\psi^*(r_2) - \psi^*(r_1)\}^{1/p} \text{ or}$$

$$\psi^*(r_2) - \psi^*(r_1) \leqslant s + \left(\frac{\psi^*(r_2)c_0 s}{\psi^*(r_1)}\right)^{1/q} \{\psi^*(r_2) - \psi^*(r_1)\}^{1/p}, \tag{iv}$$

for some constant c_0 depending on q, α, β.

Now the argument proceeds as in [2]. We let $\psi^*(r_1) > 2c_0 s$, so that the "c" in the theorem is $2c_0$. Then

$$\psi^*(r_2) - \psi^*(r_1) \leqslant s + \left\{\frac{\psi^*(r_2)}{2}\right\}^{1/q} \{\psi^*(r_2) - \psi^*(r_1)\}^{1/p},$$

and hence

$$\begin{aligned}\frac{\psi^*(r_2) - \psi^*(r_1)}{s} &\leqslant 1 + \left\{\frac{\psi^*(r_2)}{2s}\right\}^{1/q} \left\{\frac{\psi^*(r_2) - \psi^*(r_1)}{s}\right\}^{1/p} \\ &\leqslant 1 + \frac{1}{q}\frac{\psi^*(r_2)}{2s} + \frac{1}{p}\left\{\frac{\psi^*(r_2) - \psi^*(r_1)}{s}\right\}.\end{aligned}$$

Thus $\psi^*(r_2) - \psi^*(r_1) \leqslant qs + \psi^*(r_2)/2$ from which $\psi^*(r_2) \leqslant 2sq + 2\psi^*(r_1) \leqslant b\psi^*(r_1)$.

If we substitute this estimate into (iv) we obtain

$$\begin{aligned}\psi^*(r_2) - \psi^*(r_1) &\leqslant s + (cs)^{1/q}\{\psi^*(r_2) - \psi^*(r_1)\}^{1/p} \\ &\leqslant s + \frac{cs}{q} + \frac{1}{p}\{\psi^*(r_2) - \psi^*(r_1)\},\end{aligned}$$

from which we get $\psi^*(r_2) - \psi^*(r_1) \leqslant (q + c)s = ds$.

For the case $q = 1$, i.e., $p = \infty$, the restriction on α is $1 \leqslant \alpha < \infty$. From $\psi^*(r_1) \geqslant 2s$ we get

$$\int_{X \setminus S_{r_1}} f\, d\mu \leqslant \frac{cs}{\psi^*(r_1)}.$$

As before $0 \leqslant \psi^*(r_2) - \psi^*(r_1) \leqslant s + F^*(r_2) - F^*(r_1)$. Now

$$\begin{aligned}F^*(r_2) - F^*(r_1) &= \left(\int f\phi_{r_2}\, d\mu\right)^{\alpha} \psi(r_2)^{-\beta} - \left(\int f\phi_{r_1}\right)^{\alpha} \psi(r_1)^{-\beta} \\ &\leqslant \left\{\left(\int f\phi_{r_2}\right)^{\alpha} - \left(\int f\phi_{r_1}\right)^{\alpha}\right\}\psi(r_2)^{-\beta} \\ &\leqslant \alpha\left(\int f\phi_{r_2}\right)^{\alpha-1}\left\{\int f(\phi_{r_2} - \phi_{r_1})\right\}\psi(r_2)^{-\beta} \\ &\leqslant \alpha\psi(r_2)^{\alpha-1-\beta} \cdot \frac{cs}{\psi^*(r_1)}\psi(r_2) \\ &= \alpha\psi^*(r_2) \cdot \frac{cs}{\psi^*(r_1)}.\end{aligned}$$

If we now choose $\psi^*(r_1) \geqslant 2\alpha cs$, then

$$\psi^*(r_2) - \psi^*(r_1) \leqslant s + \psi^*(r_2)/2 \quad \text{or} \quad \psi^*(r_2) \leqslant b\psi^*(r_1).$$

Substitute this into the above inequality and obtain $\psi^*(r_2) - \psi^*(r_1) \leqslant ds$.

COROLLARY 1. *If* $\psi^*(r) \to \infty$ *as* $r \to \infty$, *then* $\psi^*(r) - F^*(r) \to \infty$ *as* $r \to \infty$.

REMARK. The above corollary can also be directly proven by writing $f_\infty = f' + f''$ with $\|f''\|_q \leqslant \varepsilon$ and $f' = f\chi_{S_{r_0}}$ for some large r_0, where $f_\infty = f\chi_{\cup S_r}$ and showing that $F^*(r)/\psi^*(r) \to 0$ as $r \to \infty$.

The case $\alpha = 1$, $\beta = 0$ of Theorem 2 provides us with a measurement of the amount by which Hölder's inequality misses being an equality. Let $h_f(r) = \|\phi_r\|_p - \int_X f\phi_r \, d\mu \geqslant 0$.

COROLLARY 2.

$$\max\{h_f(t), h_f(u)\} \geqslant \min\left\{\frac{\|\phi_t\|_p}{c}, \frac{\|\phi_u\|_p}{c}, \frac{\|\phi_t\|_p - \|\phi_u\|_p}{d}\right\} \equiv g(t, u).$$

PROOF. If we deny this, we have $\max\{h_f(t), h_f(u)\} \leqslant s < g(t, u)$. But then Theorem 2 tells us $\|\phi_t\|_p - \|\phi_u\|_p \leqslant ds$.

3. This paragraph is devoted to the proof of Theorem 1. We let $E_s = \{r\colon \Phi\{\psi^*(r) - F^*(r)\} \geqslant s\}$ and note that

$$\int_0^\infty \Phi\{\psi^*(r) - F^*(r)\} \, dm^* = \int_0^\infty m^*(E_s) \, ds.$$

If Ψ is the inverse of Φ, i.e., $\Psi(t) = \sup\{u\colon \Phi(u) \geqslant t\}$, then $E_s = \{r\colon \psi^*(r) - F^*(r) \leqslant \Psi(s)\}$. From Theorem 2 there are constants c, d so that, for $r_2 \geqslant r_1$, the inequalities $\psi^*(r_2) \geqslant \psi^*(r_1) \geqslant c\Psi(s)$, $\psi^*(r_j) - F^*(r_j) \leqslant \Psi(s)$, $j = 1, 2$, imply $\psi^*(r_2) - \psi^*(r_1) \leqslant d\Psi(s)$.

If $r \in E_s$ implies $\psi^*(r) \leqslant c\Psi(s)$, then the continuity of ψ^* gives $m^*(E_s) \leqslant c\Psi(s)$. If $r_2 \geqslant r_1$ are in E_s with $\psi^*(r_2) \geqslant \psi^*(r_1) \geqslant c\Psi(s)$, then $\psi^*(r_2) - \psi^*(r_1) \leqslant d\Psi(s)$, and again $m^*(E_s) \leqslant c'\Psi(s)$.

Hence, for some constant c depending on q, α, β only,

$$\int_0^\infty \Phi\{\psi^*(r) - F^*(r)\} \, dm^* \leqslant c\int_0^\infty \Psi(s) \, ds = c\int_0^\infty \Phi(u) \, du.$$

REMARK. (i) If $\psi^*(r)$ is convex on $[0, \infty)$, and hence continuous, then one can obtain

$$\int_0^\infty \Phi\{\psi^*(r) - F^*(r)\} \, dr \leqslant 2\int_0^\infty \Phi\left\{\frac{\psi^*(u)}{c}\right\} du.$$

For the proof one proceeds as in the proof of Theorem 1 with the observation that $\psi^*(r_2 - r_1) \leqslant \psi^*(r_2) - \psi^*(r_1) \leqslant d\Psi(s)$ from which $r_2 - r_1 \leqslant \psi^{*-1}(d\Psi(s))$.

(ii) Without the continuity of ψ^*, Theorem 1 need not be valid. As an example, let $X = [0, \infty)$, $\sigma(r) = \phi$, $0 \leqslant r < 1$, and $\sigma(r) = [0, 1]$, $r \geqslant 1$. If $\phi = f = \chi_{[0,1]}$, then

$\|\phi_r\|_p = 0$, $0 \leqslant r < 1$, and $\|\phi_r\|_p = 1$, $r \geqslant 1$. Hence ($\alpha = 1$, $\beta = 0$), m^* is the Dirac measure at 1, and thus

$$\int_0^\infty \Phi\{\psi^*(r) - F^*(r)\}\, dm^* = \Phi\{\psi^*(1) - F^*(1)\} = \Phi(0).$$

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