SOME CONDITIONS FOR n -CONVEX FUNCTIONS

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ABSTRACT. It is shown that if F is a function defined and continuous on $[a, b]$ such that (where n is an even integer):

(a)

$$F_{(r)}(x) = \lim_{h \rightarrow 0} \left(\frac{r!}{h^r} \right) \left(F(x+h) - F(x) - \sum_{k=1}^{r-1} \frac{h^k F_{(k)}(x)}{k!} \right)$$

exists and is finite in (a, b) for $1 < r < n-2$;

(b) for $x \in (a, b) \setminus E$, where E is countable, F is n -smooth, i.e.,

$$\lim_{h \rightarrow 0} \left(\frac{n!}{h^{n-1}} \right) \left[\frac{F(x+h) + F(x-h)}{2} - \sum_{k=0}^{n/2-1} \frac{h^{2k} D^{2k} F(x)}{(2k)!} \right] = 0,$$

where $D^{2k} F(x)$ denotes the (symmetric) de la Vallée-Poussin derivative;

(c) $\bar{D}^n F(x) > 0$ a.e. in (a, b) ;

(d) $\bar{D}^n F(x) > -\infty$ for $x \in (a, b) \setminus S$ where S is countable and $F(x)$ is n -smooth in S ; then $F(x)$ is n -convex in $[a, b]$. The same result holds for n odd. This is an improvement on the known result when S is a scattered set.

1. Introduction. The following theorem is well known (see e.g. Theorem 1.1 [2]).

THEOREM 1.1. *If F is a continuous function on (a, b) so that*

$$\bar{D}^2 F(x) = \overline{\lim}_{h \rightarrow 0} \left(\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \right) > 0 \quad \text{a.e. in } (a, b), \quad (1.1)$$

(1.2) $\bar{D}^2 F(x) > -\infty$ except on a countable set S , and

(1.3) $F(x)$ is smooth on S ,

then $F(x)$ is convex on (a, b) .

This theorem has some important applications in the theory of integrals and convergent trigonometric series (see e.g. [2] and [7]). In the investigation of trigonometric series which are summable by Cesàro means, a corresponding theorem phrased in terms of n -convex functions is required.

Several versions of such a theorem have been proved (cf. Theorem 6.1, [5] and Theorem 16, [1]), but none, so far as the author knows, in the complete generality of Theorem 1.1 insofar as the exceptional set S is concerned. The best result allows only an exceptional *scattered* set. (A set is scattered if it contains no subset that is dense in itself.)

Received by the editors July 28, 1980.

1980 *Mathematics Subject Classification.* Primary 26A51, 26A24.

Key words and phrases. Convex function.

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0002-9939/81/0000-0365/\$02.50

It is true that on the real line a scattered set is countable [6], but not every countable set is scattered (consider the rational numbers). There is some interest then in showing that an exceptional countable set may be allowed in the statement of the theorem on n -convex functions referred to above. This is the substance of the main result of this paper.

2. Notation and terminology. Let $x_0, x_1, \dots, x_n$ be $(n+1)$ distinct points from an interval $[a, b]$. By $V_n(F)$ we denote the n th divided difference of F :

$$V_n(F) \equiv \sum_{k=0}^n F(x_k)/w'_n(x_k),$$

where $w_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.

If $V_n(F) > 0$ for all choices of points $x_0, x_1, \dots, x_n$ in $[a, b]$ then F is said to be n -convex on $[a, b]$. (2-convex = convex.)

Let $F(x)$ be a real-valued function defined on the bounded interval $[a, b]$. If there exist constants $\alpha_1, \alpha_2, \dots, \alpha_r$ which depend on x_0 only and not on h , such that

$$F(x_0 + h) - F(x_0) = \sum_{k=1}^r \alpha_k \frac{h^k}{k!} + o(h^r) \quad \text{as } h \rightarrow 0, \quad (2.1)$$

then α_k , $1 \leq k \leq r$, is called the *Peano derivative of order k* of F at x_0 and is denoted by $F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0)$, $1 \leq k \leq r-1$, we write

$$\frac{h^r}{r!} \gamma_r(F; x_0, h) = F(x_0 + h) - F(x_0) - \sum_{k=1}^{r-1} \frac{h^k}{k!} F_{(k)}(x_0). \quad (2.2)$$

If there exist constants $\beta_0, \beta_2, \dots, \beta_{2r}$ which depend on x_0 and not on h , such that

$$\frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^r \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}) \quad \text{as } h \rightarrow 0,$$

then β_{2k} , $0 \leq k \leq r$, is called the *de la Vallée-Poussin derivative of order $2k$* of F at x_0 and is denoted by $D^{2k}F(x_0)$.

If F has derivatives $D^{2k}F(x_0)$, $0 \leq k \leq r-1$, we write

$$\frac{h^{2r}}{(2r)!} \theta_{2r}(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D^{2k}F(x_0)$$

and define

$$\overline{D}^{2r}F(x_0) = \lim_{h \rightarrow 0} \sup \theta_{2r}(F; x_0, h),$$

$$\underline{D}^{2r}F(x_0) = \lim_{h \rightarrow 0} \inf \theta_{2r}(F; x_0, h).$$

All the above symbols are defined similarly for odd-numbered indices (see, for example, [4, pp. 163–164]).

We denote the ordinary derivative of $F(x)$ at x_0 of order k by $F^{(k)}(x_0)$.

The function F will be said to satisfy condition A_n^* ($n \geq 3$) in $[a, b]$ if it is continuous in $[a, b]$, if, for $1 \leq k \leq n-2$, each $F_{(k)}(x)$ exists and is finite in (a, b)

and if

$$\lim_{h \rightarrow 0} h\theta_n(F; x, h) = 0 \text{ for all } x \in (a, b) \setminus E \text{ where } E \text{ is countable.} \quad (2.3)$$

(Condition A_n^* above implies condition C_n of [1].)

When a function F satisfies condition (2.3) at a point x , F is said to be n -smooth at x .

The following theorem is well-known [1]:

THEOREM 2.1. *If a function F is defined on $[a, b]$ so that*

(2.4) $F(x)$ *satisfies condition* A_n^* *in* (a, b) ,

(2.5) $\bar{D}^n F(x) \geq 0$ *a.e. in* (a, b) ;

and

(2.6) $\bar{D}^n F(x) > -\infty$ *for all* $x \in (a, b)$,

then $F(x)$ is n -convex on $[a, b]$.

3. The main result. We shall prove the following theorem:

THEOREM 3.1. *If a function F is defined on $[a, b]$ so that*

(3.1) $F(x)$ *satisfies condition* A_n^* *in* (a, b) ,

(3.2) $\bar{D}^n F(x) \geq 0$ *a.e. in* (a, b) , and

(3.3) $\bar{D}^n F(x) > -\infty$, *on* $[a, b] \setminus S$, *where* S *is countable and* $F(x)$ *is* n -*smooth in* S ,
then $F(x)$ is n -convex on $[a, b]$.

(We shall restrict the proof to the case of n even. Obvious modifications may be made to cover the case of n odd.)

We shall need the following lemmas:

LEMMA 3.1. *Given* $x_0 \in (a, b)$ *and* $\epsilon > 0$, *there exists a function* $G(x)$ *such that:*

(3.4) $\underline{D}^n G(x) \geq 0$, $x \in (a, b)$;

(3.5) $\theta_n(G; x, h) \geq 0$ *for* $x, x \pm h \in (a, b)$;

(3.6) $\lim_{h \rightarrow 0^+} h\theta_n(G; x_0, h) > 0$;

(3.7) $|D^{2k} G(x)| < \epsilon$, $x \in (a, b)$, $0 \leq k \leq n/2 - 1$;

(3.8) $G(x)$ is n -smooth if $x \neq x_0$.

PROOF. Let g be defined on $[a, b]$ by

$$g(x) = \begin{cases} x, & x \leq x_0, \\ 2x - x_0, & x \geq x_0. \end{cases}$$

Then the function G defined as a multiple of the $(n-2)$ th indefinite integral of $g(x)$ on $[a, b]$:

$$G(x) = \frac{c}{(n-3)!} \int_a^x (x-t)^{n-3} g(t) dt$$

satisfies the conditions of the lemma if $c > 0$ is chosen so that $|D^{2k} G(x)| < \epsilon$, $0 \leq k \leq n/2 - 1$. Indeed, it is clear that $\underline{D}^n G(x) \geq 0$ if $x \neq x_0$ and

$$\underline{D}^n G(x_0) \geq \underline{D}^2(D^{n-2} G(x_0)) = c \underline{D}^2 g(x_0) > 0.$$

It follows that $G(x)$ is n -convex in $[a, b]$ and (3.5) follows from lemma 4.1 [4].

Moreover

$$\begin{aligned}
 & \lim_{h \rightarrow 0+} h \theta_n(G, x_0, h) \\
 &= \lim_{h \rightarrow 0+} \left\{ \frac{n! \left[\frac{G(x_0 + h) + G(x_0 - h)}{2} - \sum_{k=0}^{n/2-1} \frac{h^{2k}}{(2k)!} D_{2k} G(x_0) \right]^{(n-2)}}{(n-1)!h} \right\} \\
 &= \lim_{h \rightarrow 0+} n \left[\frac{g(x_0 + h) + g(x_0 - h) - 2g(x_0)}{2h} \right] \\
 &= \lim_{h \rightarrow 0+} n \left[\frac{2(x_0 + h) - x_0 + x_0 - h - 2x_0}{2h} \right] = \frac{n}{2} > 0.
 \end{aligned}$$

LEMMA 3.2. If M is a function defined on $[a, b]$ such that:

(3.9) $D^{2k}M(x)$ exists and is finite in (a, b) for $1 \leq k \leq n/2 - 1$;

(3.10) $\bar{D}^n M(x) > 0$ a.e. in (a, b) ;

(3.11) $\bar{D}^n M(x) > -\infty$ on $(a, b) \setminus S$, where S is countable and $M(x)$ is n -smooth on S , then, corresponding to $\varepsilon > 0$, there is a function $Q(x)$, n -smooth for $x \notin S$, and such that $|Q(x)| < \varepsilon$ on $[a, b]$, $\bar{D}^n(M(x) + Q(x)) > 0$ a.e. on (a, b) , and

$$\bar{D}^n(M(x) + Q(x)) > -\infty$$

everywhere in (a, b) .

PROOF. Let the points of S be enumerated

$$x_1, x_2, \dots, x_n, \dots$$

and let $\{\varepsilon_i\}_{i=1}^\infty$ be a sequence such that $\sum_{i=1}^\infty \varepsilon_i < \varepsilon$. Let $M_i(x)$ be the function defined by Lemma 3.1 with ε, x_0 replaced by ε_i, x_i . Let Q be defined by

$$Q(x) = \sum_{i=1}^\infty M_i(x), \quad x \in (a, b).$$

Then $Q(x)$ is the sum of a uniformly convergent series of continuous functions and so $Q(x)$ is continuous. Also $|Q(x)| < \varepsilon, x \in (a, b)$, and $Q(x)$ is n -smooth if $x \neq x_0$. For $x \in (a, b)$, we have

$$\theta_n(Q; x, h) = \sum_{i=1}^\infty \theta_n(M_i; x, h),$$

where for each i , $\theta_n(M_i; x, h) > 0$. Therefore for each N and $x, x \pm h \in (a, b)$, we have

$$\theta_n(Q; x, h) > \sum_{i=1}^N \theta_n(M_i; x, h)$$

and

$$\bar{D}^n Q(x) = \lim_{h \rightarrow 0} \theta_n(Q; x, h) > \sum_{i=1}^N \lim_{h \rightarrow 0} \theta_n(M_i; x, h) > 0$$

for $x \in (a, b)$.

For $x_{i_0} \in S$, $h > 0$,

$$h\theta_n(Q; x_{i_0}, h) = \sum_{i=1}^{\infty} h\theta_n(M_i; x_{i_0}, h) > \sum_{i=1}^N h\theta_n(M_i; x_{i_0}, h).$$

It follows that

$$\lim_{h \rightarrow 0+} h\theta_n(Q; x_{i_0}, h) > \sum_{i=1}^N \lim_{h \rightarrow 0+} h\theta_n(M_i; x_{i_0}, h) > 0,$$

since N can be chosen so that $i_0 = N$.

Now define $F(x) = M(x) + Q(x)$. Then $\bar{D}^n F(x) > \bar{D}^n M(x) + \underline{D}^n Q(x) > 0$ a.e. in $[a, b]$, and $\bar{D}^n F(x) > \bar{D}^n M(x) > -\infty$ if $x \notin S$.

At points $x_{i_0} \in S$

$$\begin{aligned} \lim_{h \rightarrow 0+} [h\theta_n(F; x_{i_0}, h)] &> \lim_{h \rightarrow 0+} [h\theta_n(M; x_{i_0}, h)] + \lim_{h \rightarrow 0+} [h\theta_n(Q; x_{i_0}, h)] \\ &> \lim_{h \rightarrow 0+} [h\theta_n(Q; x_{i_0}, h)] > 0, \end{aligned}$$

since M is n -smooth in S . Therefore $\lim_{h \rightarrow 0+} [\theta_n(F; x_{i_0}, h)] = +\infty$.

But from the format of $\theta_n(F; x_{i_0}, h)$, with n even, this is the same as $\lim_{h \rightarrow 0-} [\theta_n(F; x_{i_0}, h)]$. Therefore $\bar{D}^n F(x_{i_0}) = \underline{D}^n F(x_{i_0}) = +\infty$.

PROOF OF THEOREM 3.1. Let $G_k(x)$ be a function corresponding to $\varepsilon = 1/k$, $k = 1, 2, \dots$, determined by Lemma 3.2. Then by Theorem 2.1 the function $M(x) + G_k(x)$ is n -convex for each k and the sequence of convex functions $\{M(x) + G_k(x)\}_{k=1}^{\infty}$ converges to the function $M(x)$ on $[a, b]$. It follows that $M(x)$ is n -convex and the proof is complete.

Since an n -convex function has continuous ordinary derivatives up to order $(n-2)$ [1], the following version of the theorem on n -convexity has some interest.

THEOREM 3.2. *If F is a function defined on $[a, b]$ such that*

- (i) $F^{(k)}(x)$ exists and is continuous, $0 < k \leq n-2$, $a < x < b$;
- (ii) $\bar{D}^n F(x) > 0$ a.e. in (a, b) ;
- (iii) $\bar{D}^n F(x) > -\infty$ except on a countable set S , and
- (iv) $F(x)$ is n -smooth in S ,

then $F(x)$ is n -convex in $[a, b]$.

PROOF. If we write

$$p(h) = F(x+h) + F(x-h) - 2 \sum_{k=0}^{n/2-1} \frac{h^{2k}}{(2k)!} F^{(2k)}(x)$$

and $q(h) = 2(h^n/n!)$, then we have (cf. [3, p. 359]):

$$\begin{aligned} \lim_{h \rightarrow 0} \left[\frac{F^{(n-2)}(x+h) + F^{(n-2)}(x-h) - 2F^{(n-2)}(x)}{h^2} \right] \\ = \bar{D}^2(D^{n-2}F(x)) > \lim_{h \rightarrow 0} \left[\frac{p(h)}{q(h)} \right] = \bar{D}^n F(x). \end{aligned}$$

It can be seen moreover by the same technique that n -smoothness of $F(x)$ implies smoothness of $D^{n-2}F(x) \equiv F^{(n-2)}(x)$. An application of Theorem 1.1 shows that $F^{(n-2)}(x)$ is convex on $[a, b]$. Thus $F^{(n-2)}(x)$ is the Lebesgue integral of a nondecreasing function, say, $p(x)$ [1]:

$$F^{(n-2)}(x) = F^{(n-2)}(a) + \int_a^x p(t) dt, \quad a \leq x \leq b. \quad (3.11)$$

We conclude then that $F(x)$ is the sum of the $(n-1)$ th integral of the nondecreasing function $p(x)$ and a polynomial of degree at most $(n-2)$. It follows [1] that $F(x)$ is n -convex on $[a, b]$.

REFERENCES

1. P. S. Bullen, *A criterion for n -convexity*, Pacific J. Math. **36** (1971), 81–98.
2. J. C. Burkill, *Integrals and trigonometric series*, Proc. London Math. Soc. (3) **1** (1951), 46–57.
3. E. W. Hobson, *The theory of functions of a real variable*, Vol. I, 3rd ed., London, 1927.
4. R. D. James, *Generalized n th primitives*, Trans. Amer. Math. Soc. **76** (1954), 149–176.
5. ———, *Summable trigonometric series*, Pacific J. Math. **6** (1956), 99–110.
6. Z. Semadeni, *Banach spaces of continuous functions*, PWN, Warsaw, 1971.
7. A. Zygmund, *Trigonometrical series*, Vol I, 2nd ed., Cambridge Univ. Press, New York, 1959.

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