MARKOV OPERATORS AND QUASI-STONIAN SPACES

ROBERT E. ATALLA

ABSTRACT. Let X be a quasi-stonian space, and let T be a σ -additive Markov operator on $C(X)$. Ando proved that if all T -invariant probabilities are σ -additive, then T is strongly ergodic (and the space of fixed points is finite-dimensional). We prove that if the set of σ -additive T -invariant probabilities is weak-* dense in the set of all T -invariant probabilities, then T is strongly ergodic. This result is easy in case X is hyperstonian. Our method of proof is to use an idea of Gordon to "hyperstonify" part of our quasi-stonian space.

1. Introduction. A compact space is *quasi-stonian* if any of the following equivalent conditions hold: (1) if A is an open F_σ set, then $\text{closure}(A)$ is clopen (= closed and open), (2) X is totally disconnected, and the algebra of clopen sets is σ -complete, (3) $C(X)$ is conditionally σ -complete. A Borel measure m on X is σ -additive if $f_n \in C(X)$, $f_1 \geq f_2 \geq \dots \geq 0$, and $\bigwedge f_n = 0$ imply $|m|(f_n) \rightarrow 0$. The operator T on $C(X)$ is σ -additive if $\bigwedge Tf_n = 0$ whenever f_n is as above [An, Section 2]. Let $\Sigma(X)$ be the Banach space of σ -additive elements of the dual space $C(X)^*$. If T is a Markov operator on $C(X)$, i.e., $T \geq 0$ and $Te = e$, where e is the unit function, let $F(T) = \{f \text{ in } C(X): Tf = f\}$, $F(T^*) = \{m \text{ in } C(X)^*: T^*m = m\}$ (where $T^* = \text{adjoint of } T$), and $P(T^*)$ be the set of probability measures in $F(T^*)$. Write $T_n = (1/n)(I + \dots + T^{n-1})$. T is called *strongly ergodic* if for each f in $C(X)$, $\lim T_n f$ exists in the Banach space $C(X)$. Our main result is

1.1. THEOREM. *If X is quasi-stonian, T a σ -additive Markov operator on $C(X)$, and $P(T^*) \cap \Sigma(X)$ is weak-* dense in $P(T^*)$, then T is strongly ergodic.*

1.2. COROLLARY [An, THEOREM 2]. *If X is quasi-stonian and T a σ -additive Markov operator on $C(X)$, the following are equivalent:*

- (a) $F(T^*) \subset \Sigma(X)$,
- (b) T is strongly ergodic, and $\dim(F(T)) = \dim(F(T^*)) < \infty$.

We note that Sato has extended Ando's theorem to include certain non-Markov positive operators [Sa, Theorem 1]. Sato's theorem is not in any obvious way a corollary to 1.1.

§2 is devoted to some general ergodic theory, and §3 to the proofs of 1.1 and 1.2.

1.3. Some concepts. A compact space is *stonian* if any of the following equivalent conditions hold: (1) if A is open, then $\text{closure}(A)$ is open, (2) X is totally

Received by the editors October 30, 1979 and, in revised form, November 20, 1979 and September 29, 1980.

1980 *Mathematics Subject Classification.* Primary 47A35; Secondary 54G05.

© 1981 American Mathematical Society
0002-9939/81/0000-0370/\$02.50

disconnected, and the Boolean algebra of clopen sets is complete, (3) $C(X)$ is a conditionally complete lattice.

To define hyperstonian spaces, we need the concept of *normal measure*, i.e., a measure m which satisfies any of the equivalent conditions: (1) m is order-continuous, i.e., if $\{f_a\}$ is a directed family in $C(X)$ and $f = \sup_a f_a$, then $\int f dm = \lim_a \int f_a dm$, (2) if F is a closed nowhere dense set, then $m(F) = 0$ [Sch, p. 149, exercise 24]. Let $\Sigma_0 = \Sigma_0(X)$ be the Banach space of all normal measures.

A compact X is *hyperstonian* if any of the following equivalent conditions hold: (1) $C(X) = (\Sigma_0)^*$, i.e., $C(X)$ is the Banach space dual of Σ_0 (where Σ_0 has its norm topology), (2) X is stonian and Σ_0 is weak-* dense in $C(X)^*$, (3) X is stonian and the union of the support sets of measures in Σ_0 is dense in X [Sch, pp. 121–122].

Clearly, a normal measure is σ -additive, as we have defined it above. In the course of proving Theorem 1.1, we shall introduce a new topology under which, as we shall prove, the σ -additive measures become normal measures.

2. Criteria for ergodicity. If R is an operator on the Banach space B , let $R^* = \text{adjoint}(R)$, $F(R) = \text{fixed points of } R$, and $F(R^*) = \text{fixed points of } R^*$. Write $R_n = (1/n)(I + \dots + R^{n-1})$. R is *strongly ergodic* if $\lim R_n x$ exists for each x in B . If R is a contraction, then R is strongly ergodic iff $F(R)$ separates $F(R^*)$, i.e., $F(R^*) \cap F(R)^\perp = (0)$ [S₂]. More generally, the separation criterion is valid if R satisfies

$$\|R_n\| < M \ (n > 1), \quad \text{and} \quad \liminf n^{-1} \|R^n\| = 0$$

[L₁, p. 123]. Note that if (1) holds for R , then it holds for R^* , R^{**} , etc.

2.1. PROPOSITION. *If R satisfies (1), the following are equivalent:*

- (a) *both R and R^* are strongly ergodic,*
- (b) $F(R^{**}) = \sigma(B^{**}, B^*)\text{-closure of } F(R)$,
- (c) $\text{norm-closure}((I - R)^*(B^*)) = \text{weak-* closure}((I - R)^*(B^*))$.

PROOF. First note that in general $\text{weak-* closure}((I - R)^*(B^*)) = F(R)^\perp$, and $\sigma(B^{**}, B^*)\text{-closure}(F(R)) = F(R)^\perp{}^\perp$.

(c) *implies* (b). In general, $F(R^{**}) = \{\text{norm-closure of } (I - R)^*(B^*)\}^\perp$, so if $\text{norm-closure}((I - R)^*(B^*)) = F(R)^\perp$, then $F(R^{**}) = F(R)^\perp{}^\perp$.

(b) *implies* (c). If (c) fails, then there exists m in $F(R)^\perp \setminus \text{norm-closure of } (I - R)^*(B^*)$. By Hahn-Banach there exists F in B^{**} with $F(m) = 1$ and $F(n - R^n) = 0$ for all n in B^* . Since F is in $F(R^{**}) \setminus F(R)^\perp{}^\perp$, (b) fails.

(c) *implies* (a). In general, $\text{norm-closure}((I - R)^*(B^*)) \cap F(R^*) = (0)$, so if (c) holds, then $F(R)^\perp \cap F(R^*) = (0)$, and hence R is strongly ergodic, by the separation theorem. Hence there exists a projection S on $F(R)$ with kernel $\text{closure}((I - R)(B))$, and then S^* yields the decomposition $B^* = F(R^*) \oplus F(R)^\perp$. But then (c) implies $B^* = F(R^*) \oplus \text{norm-closure}((I - R)^*(B^*))$, and this implies (by direct computation of averages) that R^* is strongly ergodic.

(a) *implies* (c). If R is strongly ergodic, then (as noted above) $B^* = F(R^*) \oplus F(R)^\perp$. If R^* is strongly ergodic, then $B^* = F(R^*) \oplus \text{norm-closure}((I - R)^*(B^*))$. Since $\text{norm-closure}((I - R)^*(B^*)) \subset F(R)^\perp$, we have equality.

In the next result, X will be a hyperstonian space, as defined in §1.3, with normal measures Σ_0 . Note that if R is a σ -additive Markov operator, then $R^*(\Sigma_0) \subset \Sigma_0$. Let $S = R^*|_{\Sigma_0}$. Then S is norm continuous on Σ_0 , and $S^* = R$, $S^{**} = R^*$.

2.2. PROPOSITION. *Let R be a σ -additive Markov operator on $C(X)$, where X is hyperstonian. Then R is strongly ergodic iff*

$$F(R^*) = \text{weak-}^* \text{ closure}(F(R^*) \cap \Sigma_0(X)).$$

PROOF. Necessity. If R is strongly ergodic, then there exists a projection Q such that $R_n f \rightarrow Qf$ (f in $C(X)$). Q is easily seen to be σ -additive, and hence $Q^*(\Sigma_0(X)) \subset \Sigma_0(X)$. $F(Q^*)$ is weak- * closed and Q^* is weak- * continuous, so $F(R^*) = F(Q^*) = Q^*(C(X)^*) = \text{weak-}^* \text{ closure}(Q^*(\Sigma_0(X)))$. Since $Q^*(\Sigma_0(X)) \subset \Sigma_0(X)$ and $Q^*(\Sigma_0(X)) \subset F(R^*)$, we have $F(R^*) \subset \text{weak-}^* \text{ closure}(\Sigma_0(X) \cap F(R^*))$.

Sufficiency. Put $S = R^*|_{\Sigma_0(X)}$. Then, since $R = S^*$, $R^* = S^{**}$, $C(X) = (\Sigma_0)^*$, and $C(X)^* = (\Sigma_0)^{**}$, the conclusion reads $F(S^{**}) = \sigma(\Sigma_0^{**}, \Sigma_0^*)$ -closure of $F(S)$. By “(b) implies (a)” of Proposition 2.1, it follows that S and $S^* = R$ are strongly ergodic.

3. Proofs.

3.1. Proof of Theorem 1.1. First we need some topological observations. A closed set S is a P -set if the intersection of a countable number of neighborhoods of S is again a neighborhood of S , or, equivalently, if A is an open F_σ which is disjoint from S , then $\text{closure}(A)$ is disjoint from S . By [LL₂, Lemma 1], if X is quasi-stonian and m a σ -additive regular Borel probability, then $\text{sup}(m)$ is a P -set. (By $\text{sup}(m)$ we mean its support, i.e., the smallest closed set S such that $m(S) = m(X) = 1$.) (To make the paper more self-contained, we sketch a proof. Let m be σ -additive, $S = \text{sup}(m)$, and A an open F_σ set with $A \cap S = \emptyset$. We must show $\text{closure}(A) \cap S = \emptyset$. Since X is compact and totally disconnected, $A = \bigcup A_n$, where the A_n are clopen sets. If f_n is the characteristic function of $A_1 \cup \dots \cup A_n$, then $0 = \int f_n dm$ for all n . If f is the characteristic function of the clopen set $\text{closure}(A)$, then $f = \bigvee f_n$, whence $0 = \int f dm$, and so $\text{closure}(A) \cap S = \emptyset$.) Thus the support of each m in $P(T^*) \cap \Sigma(X)$ in the hypothesis of Theorem 1.1 is a P -set. Let $K = \text{closure} \bigcup \{\text{sup}(m) : m \text{ in } P(T^*)\}$. Since $P(T^*) \cap \Sigma(X)$ is weak- * dense in $P(T^*)$, we have

$$K = \text{closure}(\bigcup \{\text{sup}(m) : m \text{ in } P(T^*) \cap \Sigma(X)\}).$$

Since X is quasi-stonian and each set in the union is a P -set, it follows from [V, Theorem 2] that K is also a P -set. (Again, we sketch a proof. Let $K = \text{closure}(\bigcup K_\alpha)$, where the K_α are P -sets, and A an open F_σ with $A \cap K = \emptyset$. Suppose $\text{closure}(A) \cap K \neq \emptyset$. Since $\text{closure}(A)$ is open, it follows that $\text{closure}(A) \cap K_\alpha \neq \emptyset$ for some index α . But $A \cap K_\alpha = \emptyset$ and K_α is a P -set, so we have a contradiction.)

Now K is a T -invariant set, i.e., $f|K = 0$ implies $Tf|K = 0$ [S₁, Theorems 1.1, 1.3]. Hence T induces in a natural way an operator T_0 on $C(K)$. Namely if g is in

$C(K)$ and x is in K , let $\bar{g}$ in $C(X)$ be an extension of g , and let $T_0 g(x) = T\bar{g}(x) = \int \bar{g} d(T^* \delta_x)$. Since K is a P -set, [A₂, Proposition 3.5] implies that if T_0 is strongly ergodic, then so is T . Thus it suffices to prove that the restricted operator T_0 is strongly ergodic.

Imitating a construction of Gordon [G], let

$$Y = \bigcup \{ \text{sup}(m) : m \text{ in } P(T^*) \cap \Sigma(X) \}.$$

We denote the original topology of X (and hence of Y and $\text{closure}(Y) = K$) by the name α . We shall show later that if f is in $C(X)$, then $Tf_n|_Y$ converges uniformly on Y , hence on $K = \text{closure}(Y)$.

Following [G, Section 5], we introduce a new topology δ on Y , having as base of open sets all sets of the form $X_m = \text{sup}(m)$, where m is any σ -additive measure whose support is contained in Y . We show that $X_m \cap X_p = X_q$ for some q in $\Sigma(X)$ as follows: since X_m is a P -set and X_p a support set, $X_m \cap X_p$ is α -clopen in X_p . If f is the characteristic function of $X_m \cap X_p$, put $dq = fdp$ [A₁, Theorem 1]. Clearly q is σ -additive. Moreover since $X_m \cap X_p$ is α -open in X_p for each X_m , δ is a weaker topology on X_p than α is. Since δ is Hausdorff and (X_p, α) is compact, (X_p, δ) is compact and $\alpha = \delta$ on X_p . Thus, each X_p is open and compact in the δ -topology. (To show δ is Hausdorff, let $x \neq y$ in X_p , and find A, B clopen and disjoint in (X_p, α) such that x is in A and y is in B . Then $\chi_A dp$ and $\chi_B dp$ define members of Σ with disjoint supports containing x and y respectively.)

Let $\tilde{Y}$ be the Stone-Čech compactification of (Y, δ) . To show that $\tilde{Y}$ is hyperstonian we depart from Gordon's procedure [G, Section 6] and show (i) Y , and hence $\tilde{Y}$, is extremally disconnected, and (ii) the union of the supports of normal measures is dense in $\tilde{Y}$ [Sch, p. 121].

For (i), we shall be using the δ -topology except where otherwise mentioned. Let V be open and let x be in $\text{closure}(V)$. We must show x is in $\text{interior}(\text{closure}(V))$. Fix an X_m containing x . Then x is in $V \cap X_m$, which is an open subset of X_m . If X_p is any basic neighborhood of x , $X_p \cap (V \cap X_m)$ contains x and is an open subset of $V \cap X_m$, and hence x is interior to $V \cap X_m$ in the subspace topology of X_m . But on X_m , $\alpha = \delta$, and X_m is extremally disconnected for α [Se, Theorem 2.2], so x is interior to the closure in X_m of $V \cap X_m$. Hence there exists a basic open set X_q with $x \in X_q \subset X_m$ and $X_q \subset X_m\text{-closure}(V \cap X_m) \subset Y\text{-closure}(V)$. Thus, x is interior to $\text{closure}(V)$.

For (ii) we show that if m is in $P(T^*) \cap \Sigma(X)$, then with respect to the δ -topology on $\tilde{Y}$ it becomes a normal measure. As noted in §1.3, it suffices to show that if F is a δ -closed nowhere dense set, then $m(F) = 0$. Let $S = X_m$, a δ -clopen set. Since S is open, $F \cap S$ is closed and nowhere dense in the relative δ -topology of S ; and since the relative α and δ -topologies coincide on S , $F \cap S$ is closed and nowhere dense in the relative α -topology of S . We now show that there exists in X an α -closed G_δ set W , α -nowhere dense in X , such that $F \cap S \subset W \cap S$. Let V_n be α -open sets with $F \cap S \subset V_n$ and $m(F) = m(F \cap S) = \lim m(V_n)$. Since $F \cap S$ is α -closed, hence α -compact, we may assume each V_n is α -clopen, and $Z = \bigcap V_n$ is an α -closed G_δ set with $F \cap S \subset Z$ and $m(Z \setminus F) = 0$. Now $Z \cap S$ is

α -nowhere dense in S . (For otherwise there exists α -clopen B with $\emptyset \neq B \cap S \subset Z \cap S$. $B \cap S \not\subset F \cap S$, since $F \cap S$ is α -nowhere dense in S , so $(B \setminus F) \cap S \neq \emptyset$, whence $m(B \setminus F) > 0$. But $(B \setminus F) \cap S \subset (Z \setminus F) \cap S$, so $m(Z \setminus F) > 0$, a contradiction.) Since X is quasi-stonian, the set $A = \alpha\text{-interior}(Z)$ is clopen. Since $Z \cap S$ is nowhere dense in S , $A \cap S = \emptyset$, so $W = Z \setminus A$ is a nowhere dense α -closed G_δ with $W \cap S = Z \cap S \supset F \cap S$.

Finally we show $m(W) = 0$, and hence $m(F) = 0$. But W has the form $W = \bigcap A_n$, where the A_n are α -clopen, $A_1 \supset A_2 \supset \dots$. If f_n is the characteristic function of A_n , then since $\text{interior}(W) = \emptyset$, we have $\bigwedge f_n = 0$, and so $m(W) \leq \lim \int f_n dm = 0$. (ii) is proved.

We now show that our original operator T induces an operator S on $C(\tilde{Y})$. If m is in $P(T^*) \cap \Sigma(Y)$, then for x in X_m we have $\text{supp}(T^*\delta_x) \subset X_m$. If f is a δ -continuous function on $\tilde{Y}$ and x in X_m , $f|_{X_m}$ is α -continuous on X_m , and we define $Sf(x) = \int f d(T^*\delta_x)$. Then Sf is defined on all $\tilde{Y}$ as the Stone-Čech extension.

We complete the proof by showing that S is uniformly ergodic, whence, if f is in $C(K)$, then $T_n f|_Y$ converges uniformly, and hence so does $T_n f|_K$. To apply Proposition 2.2, we must prove $P(S^*) \cap \Sigma_0(Y)$ is weak-* dense in $P(S^*)$. (Note that the δ -topology introduces no new regular Borel measures on Y . For if m is one whose support is contained in Y , then since each X_p is a δ -clopen set, $m(X_p) > 0$ whenever $\text{supp}(m) \cap X_p \neq \emptyset$. Hence $\text{support}(m)$ is contained in a countable collection of the X_p sets. The α -closure of the union of this collection is contained in Y , and is also an X_p set: if $X_{p(i)}$ are the sets in question, just let $p = \Sigma 2^{-i} p(i)$.)

Suppose $P(S^*) \cap \Sigma_0(Y)$ is not weak-* dense in $P(S^*)$, and let m_0 be in $P(S^*)$, but not in ${}^*\text{-closure}(P(S^*) \cap \Sigma_0(Y))$. By separation [D, p. 22], there exists f in $C(Y)$ such that $p(f) < -1$ for all p in $P(S^*) \cap \Sigma_0(Y)$, and $m_0(f) > 0$. Now $\{S_n f: n \geq 1\}$ is a norm bounded set in $C(Y)$, and since $C(\tilde{Y})$ is the dual of $\Sigma_0(Y)$, it is precompact. Let g in $C(Y)$ be a $\sigma(\Sigma_0(Y), C(\tilde{Y}))$ cluster point, and let $\{S_{n(i)}: i \text{ in } I\}$ be a subnet such that $\int S_{n(i)} f dp \rightarrow \int g dp$ for all p in $\Sigma_0(Y)$. Hence $\int g dm_0 > 0$ and $\int g dp < -1$ for all p in $\Sigma_0(Y)$. Further, $Sg = g$. (Proof: $\|S_{n(i)} Sg - S_{n(i)} g\| \leq n(i)^{-1} \|S^{n(i)} g - g\| \leq 2n(i)^{-1} \|g\| \rightarrow 0$, whence $\int Sg dp = \int g dp$ for all p in $\Sigma_0(Y)$. Since, as noted above, $\Sigma_0(\tilde{Y})$ is weak-* dense in $C(\tilde{Y})^*$, we have $Sg = g$.) If p is an extreme point of $P(S^*)$, then g is constant on $\text{supp}(p)$ [S₁, Theorem 1.11]. If, further, $\text{supp}(p) \subset X_m$ for some m in $P(S^*) \cap \Sigma_0(Y)$, then since by hypothesis $P(T^*) \cap \Sigma(Y)$ is dense in $P(T^*)$ and the α -topology on X_m is the same as the δ -topology, it follows that $p(g) \leq -1$ and hence $g \leq -1$ on $\text{supp}(p)$. By Krein-Milman, m is in the weak-* closed convex hull of the extreme points of $P(S^*)$, and it is elementary to show that these extreme points may be assumed to have supports contained in $\text{supp}(m) = X_m$. Hence

$$X_m = \text{closure}(\bigcup \{\text{supp}(p): p \text{ extreme, } \text{supp}(p) \subset X_m\}),$$

whence $g \leq -1$ on X_m . It follows that $g \leq -1$ on Y , and, by density, on $\tilde{Y}$. But this contradicts $m_0(g) > 0$. This proves $P(S^*) \cap \Sigma_0(Y)$ is * -dense in $P(S^*)$, and we are through.

3.2. Proof of the corollary.

(a) *implies* (b). If (a), then the theorem implies that T is strongly ergodic, and we must show that $\dim F(T) < \infty$. Since by [S₁, Theorem 1.11], $\dim F(T) < \text{number of extreme points in } P(T^*)$, it suffices to prove that the latter set is finite. Suppose $m_1, m_2, \dots$ are distinct extreme points, and let $S_i = \text{sup}(m_i)$. Since T is strongly ergodic, the sets S_i are pairwise disjoint. (By [S₁, Theorem 2.7], invariant functions separate the m_i , and by [S₁, Theorem 1.11], an invariant function is constant on each S_i .) Since $P(T^*) \subset \Sigma(X)$, the S_i are P -sets, and hence a routine induction gives a pairwise disjoint sequence of clopen sets A_i with $S_i \subset A_i$. Define f_i in $C(X)$ to be the characteristic function of $B_i = \text{closure } \bigcup_{n \geq i} A_n$. Then $\bigwedge f_n = 0$, while if m is a weak-* cluster point of $\{m_i\}$, then $\int f_n dm = 1$ for all n . Thus m is an element of $P(T^*)$ which is not σ -additive, contrary to hypothesis.

(b) *implies* (a). This is the easy part; see [An].

ADDED IN PROOF. In his thesis, D. Axmann gives a treatment of Ando's condition in the context of vector lattices, and shows that under this condition T is actually ergodic in the uniform operator topology (*Struktur und Ergodentheorie irreduzibler operatoren auf Banachverbänden*, Dissertation, Eberhard-Karls-Universität zu Tübingen, 1980, Chapter 4). I do not know whether our condition implies uniform ergodicity.

REFERENCES

- [An] T. Ando, *Positive Kontraktionen in $C(X)$* , *Studia Math.* **31** (1968), 173–187.
- [A₁] R. Atalla, *Measure theoretic behavior of closed sets*, *Comm. Math. Univ. Carolinae* **19** (1978), 697–703.
- [A₂] ———, *Generalized almost-convergence vs. matrix summability*, *Colloq. Math.* (to appear).
- [D] M. Day, *Normed linear spaces*, 1st ed., Springer, Berlin and New York, 1962.
- [G] H. Gordon, *The maximal ideal space of a ring of measurable functions*, *Amer. J. Math.* **88** (1966), 827–843.
- [L₁] S. Lloyd, *On the mean ergodic theorem of Sine*, *Proc. Amer. Math. Soc.* **56** (1976), 121–126.
- [L₂] ———, *Two lifting theorems*, *Proc. Amer. Math. Soc.* **42** (1974), 128–134.
- [Sa] R. Sato, *Invariant measures and ergodic theorems for positive operators on $C(X)$ with X quasi-stonian*, *Math. J. Okayama Univ.* **22** (1979), 77–90.
- [Sch] H. Schaefer, *Banach lattices and positive operators*, Springer, Berlin and New York, 1974.
- [Se] G. Seever, *Measures on F -spaces*, *Trans. Amer. Math. Soc.* **133** (1968), 267–280.
- [S₁] R. Sine, *Geometric theory of a single Markov operator*, *Pacific J. Math.* **27** (1968), 155–166.
- [S₂] ———, *A mean ergodic theorem*, *Proc. Amer. Math. Soc.* **24** (1970), 438–439.
- [V] A. Veksler, *P -sets in topological spaces*, *Dokl. Akad. Nauk SSSR* **193** (1970) = *Soviet Math. Dokl.* **11** (1970), 953–956.

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701