PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

THE FRÉCHET SPACE ω ADMITS A STRICTLY STRONGER SEPARABLE AND QUASICOMPLETE LOCALLY CONVEX TOPOLOGY

SUSANNE DIEROLF

Let $\mathfrak{K}$ denote the class of all locally convex Hausdorff spaces $(E, \mathfrak{T})$ with the following property: Every locally convex Hausdorff topology $\mathfrak{S} \subset \mathfrak{T}$ on E has the same subfamily summable sequences as $\mathfrak{T}$. Several articles have been devoted to the investigation of the richness of $\mathfrak{K}$, e.g., Kalton **[4]**, Labuda **[6]**, **[7]**, Graves **[3]**; see also the references in **[3]**. For example, $\mathfrak{K}$ contains every fully complete locally convex space which does not contain l^∞ **[6**, p. 219, (8)], hence every separable Fréchet space. E. Thomas asked in a letter of 1976 whether $\mathfrak{K}$ even contains every separable quasicomplete space. This note provides a negative answer to this question.

We will use the following results about separability which we prove for general topological vector spaces.

LEMMA. *Every finite codimensional linear subspace H in a separable topological vector space E is separable.*

PROOF. We may at once assume that $H = \ker f$, where f is a discontinuous linear form on E.

E contains a dense linear subspace L of countable dimension. For every $x \in E$ let L_x denote the linear span of $L \cup \{x\}$. We denote the topology of E by $\mathfrak{T}$. The strongest linear topology $\mathfrak{S}$ on E such that for every $x \in E$, the relative topologies $\mathfrak{S}|L_x$ and $\mathfrak{T}|L_x$ coincide, is clearly stronger than $\mathfrak{T}$. Moreover $\mathfrak{S}|L = \mathfrak{T}|L$ and L is dense in $(E, \mathfrak{S})$, hence $\mathfrak{T} = \mathfrak{S}$ by **[2**, p. 349, Lemma 1]. Since f is discontinuous we deduce that for some $z \in E$ the restriction $f|L_z$ is discontinuous, whence $H \cap L_z$ is dense in L_z. Thus $H \cap L_z$ is dense in E and hence dense in H. Since $H \cap L_z$ is of countable dimension, we have proved that H is separable. $\square$

(For a locally convex space E, a somewhat technical proof of the lemma has been given by Valdivia in **[8**, p. 195, Lemma 2].)

PROPOSITION. *Let $(E, \mathfrak{T})$ be a separable topological vector space over $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$ and let $(f_n)_{n \in \mathbf{N}}$ be a sequence of linear forms on E. Then the initial topology $\mathfrak{F}$ on E with respect to the identity map* id: $E \to (E, \mathfrak{T})$ *and all the functionals f_n: $E \to \mathbf{K}$ ($n \in \mathbf{N}$) is again separable.*

Received by the editors September 29, 1980.
1980 *Mathematics Subject Classification.* Primary 46A35, 46A05.
Key words and phrases. Separable topological vector spaces, subseries convergent series, Orlicz-Pettis property.

PROOF. For every $n \in \mathbf{N}$, the space E provided with the initial topology $\mathfrak{F}_n$ with respect to id: $E \to (E, \mathfrak{T})$ and f_i: $E \to \mathbf{K}$ $(1 \leq i \leq n)$, is the topological direct sum of $(\bigcap_{1 \leq i \leq n} \ker f_i, \mathfrak{T}| \bigcap_{1 \leq i \leq n} \ker f_i)$ and a finite dimensional linear subspace, hence separable according to the lemma. Since $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$ $(n \in \mathbf{N})$ and $\mathfrak{F}$ equals the supremum $\bigvee_{n \in \mathbf{N}} \mathfrak{F}_n$, we obtain the separability of $(E, \mathfrak{F})$. □

The separable Fréchet space $\omega := \mathbf{K}^{\mathbf{N}}$ provided with the product topology $\mathfrak{P}$, clearly carries the initial topology with respect to the sequence of linear forms p_n: $\omega \to \mathbf{K}$, $(x_m)_{m \in \mathbf{N}} \mapsto x_n$, $(n \in \mathbf{N})$. Thus we get the following:

COROLLARY. *For every separable linear topology* $\mathfrak{T}$ *on* ω *the supremum* $\mathfrak{T} \vee \mathfrak{P}$ *is again separable.*

REMARK. We mention that the supremum of two separable linear topologies need not be separable. In fact, let $(E, \mathfrak{T})$ be a separable locally convex space containing a nonseparable linear subspace L. Choose a linear subspace $M \subset E$ such that $L \cap M = \{0\}$ and $L + M = E$. Then the initial topology $\mathfrak{S}$ on E with respect to j: $E \to (E, \mathfrak{T})$, $j(x + y) := x - y$ $(x \in L, y \in M)$ is also separable. One verifies without difficulty that $(E, \mathfrak{T} \vee \mathfrak{S})$ is the topologically direct sum of $(L, \mathfrak{T}|L)$ and $(M, \mathfrak{T}|M)$, hence not separable.

EXAMPLE. We consider the noncomplete separable Montel space X constructed by Amemyia, Kōmura [1] (cf. also Knowles, Cook [5]), whose dimension is not less than the dimension of ω and in which every bounded subset has a finite dimensional linear span (see [1], [5]). Consequently there exists an injective linear map f: $\omega \to X$ with separable range. Let $\mathfrak{T}$ denote the initial topology on ω with respect to f: $\omega \to X$, which is clearly locally convex.

On account of the corollary, $(\omega, \mathfrak{T} \vee \mathfrak{P})$ is separable. Moreover, every bounded set in $(\omega, \mathfrak{T} \vee \mathfrak{P})$ has finite dimensional linear span, whence in particular, $(\omega, \mathfrak{T} \vee \mathfrak{P})$ is quasicomplete.

Finally, the sequence $(e_n)_{n \in \mathbf{N}}$ of unit vectors $e_n = (\delta_{nm})_{m \in \mathbf{N}} \in \omega$ is subfamily summable in $(\omega, \mathfrak{P})$, but not bounded, hence not summable, in $(\omega, \mathfrak{T} \vee \mathfrak{P})$. Thus $(\omega, \mathfrak{T} \vee \mathfrak{P}) \notin \mathfrak{L}$.

REFERENCES

1. I. Amemyia and Y. Kōmura, *Über nicht vollständige Montelräume*, Math. Ann. **177** (1968), 273–277.

2. S. Dierolf and U. Schwanengel, *Examples of locally compact non-compact minimal topological groups*, Pacific J. Math. **82** (1979), 349–355.

3. W. H. Graves, *Universal Lusin measurability and subfamily summable families in abelian topological groups*, Proc. Amer. Math. Soc. **73** (1979), 45–50.

4. N. J. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. **10** (1971), 402–412.

5. R. J. Knowles and T. A. Cook, *Incomplete reflexive spaces without Schauder bases*, Proc. Cambridge Philos. Soc. **74** (1973), 83–86.

6. I. Labuda, *A note on exhaustive measures*, Ann. Soc. Math. Polon. Ser. I Comment. Math. Prace Mat. **18** (1975), 217–221.

7. ______, *Universal measurability and summable families in tvs*, Indag. Math. **41** (1979), 27–34.

8. M. Valdivia, *On quasi-completeness and sequential completeness in locally convex spaces*, J. Reine Angew. Math. **276** (1975), 190–199.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN, D-8000 MÜNCHEN 2, WEST GERMANY