A CHARACTERIZATION OF C -COMPACT SPACES

JESSE P. CLAY AND JAMES E. JOSEPH

ABSTRACT. a characterization of C -compact spaces is given in terms of relations.

The class of C -compact spaces has been of continuing interest to a significant number of researchers (see [C] and the list of references for [H-L₃] and [L-T]). The purpose of this note is to give a characterization of C -compact spaces in terms of certain relations. This characterization analogizes the characterizations of H -closed and minimal Hausdorff spaces given by Herrington and Long [H-L₁], [H-L₂] in terms of graphs of functions into the spaces. The study of relations has also received and continues to receive wide attention from researchers (see the list of references for [S]). All spaces will be assumed to be Hausdorff and if A is a subset of a space we will denote the family of open neighborhoods of A by $\Sigma(A)$ ($\Sigma(x)$ if $A = \{x\}$). For $x \in X$ we will let $\Lambda(x) = \{V - \{x\} : V \in \Sigma(x)\}$. If $R \subset X \times Y$ is a relation we will assume that the domain of R is X . If $A \subset X$ we write $R[A]$ ($R[x]$ if $A = \{x\}$) for $\{y \in Y : (x, y) \in R \text{ for some } x \in A\}$ and if Ω is a family of subsets of X we write $R[\Omega]$ for $\{R[A] : A \in \Omega\}$.

The θ -closure of a subset A of a space is the intersection of the closures of the members of $\Sigma(A)$ and the θ -adherence of a filterbase Ω on a space is the intersection of the θ -closures of the members of the filterbase. A space is C -compact if each closed subset A of the space and filterbase Ω on A satisfy $A \cap \text{ad}_\theta \Omega \neq \emptyset$ [H-L₃]. If X and Y are spaces, a relation $R \subset X \times Y$ is *strongly-subclosed* if $\text{ad}_\theta R[\Lambda(x)] \subset R[x]$ for each $x \in X$ for which $\Lambda(x)$ is a filterbase on X . R is *strongly-closed* if R is a strongly-subclosed and $R[x]$ is θ -closed for each $x \in X$. R is *upper-semicontinuous* (u.s.c.) if for each $x \in X$ and open W containing $R[x]$, there is a V open about x satisfying $R[V] \subset W$. Herrington and Long introduced the notion of strongly-closed graph for functions in [H-L₁] and showed in [H-L₂] that a space is minimal Hausdorff if and only if all functions with strongly-closed graphs into the space are continuous.

We prove the following similar result for C -compact spaces.

THEOREM. *A space Y is C -compact if and only if for all spaces X , all strongly-subclosed relations $R \subset X \times Y$ are u.s.c.*

PROOF. We first prove the necessity. Let Y be C -compact, let X be a space and let $R \subset X \times Y$ be strongly-subclosed. If $x \in X$ and some $W \in \Sigma(R[x])$ satisfies

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$\emptyset \neq R[V] - W$ for all $V \in \Sigma(x)$, then $\Lambda(x)$ is a filterbase on X and $\emptyset \neq \text{ad}_\theta R[\Lambda(x)] - W \subset R[x] - W$ since Y is C -compact and R is strongly-subclosed. This is a contradiction. For the sufficiency, suppose that A is a nonempty closed subset of Y and that Ω is a filterbase on A satisfying $A \cap \text{ad}_\theta \Omega = \emptyset$. We know from Theorem 10 of [H-L₁] that $A \neq Y$. Choose $x_0 \in A$ and let $X = Y$ endowed with the topology $\{V \subset X: x_0 \notin V \text{ or some } F \in \Omega \text{ satisfies } F \subset V\}$. Let $R = \{(x, x): x \neq x_0\} \cup \{(x_0, y): y \in Y - A\}$. It is not difficult to show that X is Hausdorff. To see that R is strongly-subclosed we need check only at x_0 . $\Omega \subset \Lambda(x_0)$ so $\text{ad}_\theta R[\Lambda(x_0)] \subset \text{ad}_\theta R[\Omega] = \text{ad}_\theta \Omega \subset Y - A = R[x_0]$. Hence R is strongly-subclosed. However R is not u.s.c. since $Y - A \in \Sigma(R[x_0])$ and no $V \in \Sigma(x_0)$ satisfies $R[V] \subset Y - A$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, D.C. 20059