KRULL DIMENSION AND REFLEXIVITY IN SOME NOETHERIAN RINGS

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ABSTRACT. In this paper we study Noetherian prime rings R which satisfy the formula $|R| = \sup\{|I^{**}/I| : I \text{ is an essential left ideal of } R\} + 2$, where $||$ denotes left Krull dimension. If further Q/R is $|R| - 1$ unmixed, where Q is the simple Artinian quotient ring of R , we characterize R using torsion theories cogenerated by the injective hulls of $|R| - 1$ dimensional critical modules. Also equivalent statements are established, linking homological properties with dimension theory, for R -modules to be reflexive.

Introduction. Let R be a left and right Noetherian ring. Then R has left and right Krull dimension [1]. Denoting (left) Krull dimension by $||$, it is well known that if further R is prime, then $|R| \leq s(R) + 2$ where $s(R) = \sup\{|I^{**}/I| : I \text{ is essential in } R\}$ and $I^{**} = \text{Hom}_R(\text{Hom}_R(I, R), R)$. Commutative domains which satisfy $|R| = s(R) + 2$ are classified in [6]. Some extensions to noncommutative rings are considered in §1. Adding the condition that $R = \bigcap R_\tau$, where τ ranges over torsion theories cogenerated by the injective hulls of $|R| - 1$ dimensional critical modules, we characterize reflexive modules over R . In particular we show that in such rings an essential ideal I of R is reflexive if and only if the Krull dimension of any nonzero submodule of R/I is $|R| - 1$. In §3 we consider prime rings R satisfying $R = \bigcap R_\tau$ where each R_τ is a hereditary ring.

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Notations. For an R -module M we denote $\text{Hom}_R(M, R)$ by M^* . Let R be an order in a simple Artinian quotient ring Q and let I be a left R -submodule of Q such that $I \cap R$ is essential. Then by [6, IX, 5.1]

$$I^* \simeq I^{-1} = \{q \in Q | Iq \subseteq R\}.$$

A similar result holds for right submodules. Identifying I^* with I^{-1} we have $I \subseteq I^{**}$ and $I^* = I^{***}$. Further I is reflexive if and only if $I = I^{**}$.

All rings have a 1 and all modules are unital. Unless otherwise specified, all properties are left properties. A regular ideal is a *left* ideal with a regular element. The Krull dimension is denoted by $||$.

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For the rest of this paper R is a prime left and right Noetherian ring. Denote the quotient ring of R by Q and $|R|$ by α . We make the assumption that α is not a limit ordinal. A *height one localisation* τ over R denotes localisation at a torsion theory cogenerated by the injective hull of a (nonzero) $\alpha - 1$ dimensional cyclic critical R -module. Following [6] we write R_τ , M_τ for the localisations of R as a left module and arbitrary left R -modules M respectively.

§1

DEFINITION 1.1. A ring R has (M2) if $\sup\{|I^{**}/I|: I \text{ is essential in } R\} = \alpha - 2$.

LEMMA 1.2. Let M be an R -module with Krull dimension. Then $|M| \leq \alpha - 2$ if and only if $M_\tau = 0$ for every height one localisation τ .

PROOF. If $|M| \leq \alpha - 2$ then $\text{Hom}(M, E(A)) = 0$ for criticals A of dimension $\alpha - 1$ and hence $M_\tau = 0$ for every τ of height one. Conversely if $|M| \geq \alpha - 1$, there is an element m in M with $|Rm| \geq \alpha - 1$, and since Rm is Noetherian it has an $\alpha - 1$ critical A as a homomorphic image. If τ is cogenerated by $E(A)$ it is easily verified that $M_\tau \neq 0$.

LEMMA 1.3. Let I be an ideal of R and τ a height one localisation. If I is τ -torsion then I is essential. Consequently $R_\tau = \{q \in Q: q \in J^* \text{ for some } \tau\text{-torsion ideal } J\}$.

PROOF. Suppose there is a nonzero ideal K in R with $I \cap K = 0$. Since R is prime KR contains a regular element. Hence $\text{Hom}(K^{(n)}, E(A)) \neq 0$ for some n , and so $\text{Hom}(K, E(A)) \neq 0$. It follows that $\text{Hom}(R/I, E(A)) \neq 0$.

PROPOSITION 1.4. Let R have (M2) and suppose that $R_\tau I = I_\tau$ for every essential ideal I of R and every height one localisation τ . Then every ideal of R_τ is reflexive.

PROOF. From the exact sequence $0 \rightarrow I_\tau \rightarrow (I^{**})_\tau \rightarrow (I^{**}/I)_\tau$ we get, using (1.2), that $I_\tau = (I^{**})_\tau$. Clearly $I^* R_\tau \subseteq (R_\tau I)^*$; hence $(I_\tau)^{**} \subseteq (I^* R_\tau)^*$. Let $q \in (I^{**})_\tau$. Then there is a τ -torsion ideal J with $Jq \subseteq I^{**}$; hence $JqI^* \subseteq R$, that is, $qI^* \subseteq R_\tau$. Thus $q \in (I^* R_\tau)^*$. Since the argument is reversible we have $(I^* R_\tau)^* = (I^{**})_\tau$. Therefore $(I_\tau)^{**} \subseteq (I^{**})_\tau = I_\tau$, and so I_τ is reflexive. Now if T is an essential ideal of R_τ , then $T \cap R$ is essential in R and $R_\tau(T \cap R) \subseteq T \subseteq (T \cap R)_\tau$. Hence $T = (T \cap R)_\tau$, and by the above T is reflexive.

LEMMA 1.5. Suppose that every ideal of R_τ is reflexive for every height one localisation τ . Then $R_\tau I = I_\tau$ for every ideal I of R .

PROOF. It suffices to prove the lemma for essential ideals I . Clearly $(I_\tau)^* \subseteq (R_\tau I)^*$. Let $q \in (R_\tau I)^*$. Then q induces a homomorphism $I_\tau/R_\tau I \rightarrow Q/R_\tau$. Since $I_\tau/R_\tau I$ is τ -torsion and Q/R_τ is τ -torsion free q must induce the zero map; hence $q \in (I_\tau)^*$. Thus $(I_\tau)^* = (R_\tau I)^*$, and so $(I_\tau)^{**} = (R_\tau I)^{**}$, that is $I_\tau = R_\tau I$.

THEOREM 1.6. Consider the following statements.

- (i) R has (M2) and $R_\tau I = I_\tau$ for every ideal I of R and every height one τ .
- (ii) Every ideal of R_τ is reflexive for every height one τ .

Then (i) $\Rightarrow$ (ii). If each R_τ is left and right R -flat then (ii) $\Rightarrow$ (i).

PROOF. That (i) implies (ii) is by (1.4). Now suppose that each R_τ is flat on both sides and show that (ii) $\Rightarrow$ (i). From (1.5) it remains to prove that R has (M2). Let $F \rightarrow G \rightarrow I \rightarrow 0$ be a free resolution of an essential ideal I with F, G finitely generated. There exists a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & I^* \otimes R_\tau & \rightarrow & G^* \otimes R_\tau & \rightarrow & F^* \otimes R_\tau \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ & & (R_\tau \otimes I)^* & \rightarrow & (G_\tau)^* & \rightarrow & (F_\tau)^* \end{array}$$

with exact rows since R_τ is flat. It follows that $I^* \otimes R_\tau \simeq (R_\tau \otimes I)^*$. Since R_τ is right R -flat $R_\tau \otimes I \simeq R_\tau I$; hence $I^* \otimes R_\tau \simeq (I_\tau)^*$. Thus $(I^* \otimes R_\tau)^* \simeq (I_\tau)^{**}$ by (ii). Using $I^* = I^{***}$ we thus obtain $(I^{**})_\tau = (I_\tau)^{**} = I_\tau$ and hence R has (M2) by (1.2).

The next theorem shows that the conditions of (1.6) are equivalent for fully bounded classical prime rings. See [6, VII, 4.4].

PROPOSITION 1.7. *Let τ be a height one localisation, C a subset of R satisfying: an ideal I is τ -torsion if and only if $I \cap C \neq \emptyset$. Then any element of C is regular and $R_\tau = C^{-1}R$. Consequently R_τ is right R -flat.*

PROOF. For x in C the ideal Rx is τ -torsion and so, by (1.3), x is regular. Hence x is invertible in Q and clearly x^{-1} is in R_τ . If $q \in R_\tau$ then $Iq \subseteq R$ for some τ -torsion ideal I . Hence there is an element c in $I \cap C$ such that $q = c^{-1}r$ for some $r \in R$.

We need the following result which is immediate from [4, 2.3].

PROPOSITION 1.8. *Let S be a left and right Noetherian left and right fully bounded ring. Let M be an S -bimodule, finitely generated on both sides. Let I, J be the left and right annihilators of M respectively. Then $|S/I| = |M| = |S/J|$.*

THEOREM 1.9. *Let R be a left and right classical left and right fully bounded prime ring. Then R_τ is left and right R -flat for every height one localisation τ . Consequently the conditions of (1.6) are equivalent for R .*

PROOF. Let τ be a height one torsion theory cogenerated by $E(A)$, where A is an $\alpha - 1$ critical. Let $P = \text{Ass}(A)$. Then $|R/P| = |A|$ and τ is also the torsion theory cogenerated by $E(R/P)$ [6, VII, §2]. Then $\text{Hom}(R/I, E(A)) = 0$ implies that $\text{Hom}(R/I + P, E(R/P)) = 0$ and this is possible only if $(I + P)/P$ is essential in R/P . Let $C(\tau) = \{c \in R: (Rc + P)/P \text{ is essential in } R/P\}$. Then from the above, I is τ -torsion implies that $I \cap C(\tau) \neq \emptyset$. Conversely, suppose $Rc + P/P$ is essential in R/P . Then since R is fully bounded, $P \subseteq J \subseteq Rc + P$ for some two-sided ideal J . Hence, if $I_n = (Rc + P^{n+1}) \cap P^n$, we have $P^n J \subseteq I_n$, and hence that $|P^n/I_n| < |P^n/P^n J| < |R/J|$ by (1.8), and so $|P^n/I_n| < |R/J| < |R/P| = |A|$. An induction now yields $|R/(Rc + P^n)| < |R/P|$. If $cq = 0$ for some $q \in E(A)$ then since $P^n q = 0$ for some n , we get $|Rq| < |A|$ which yields $q = 0$. Hence $\text{Hom}(R/Rc, E(A)) = 0$, that is Rc is τ -torsion. Furthermore analogous arguments show that if σ is the right torsion theory cogenerated by the injective hull of R/P

we get T is σ -torsion if and only if $T \cap C(\sigma) \neq \emptyset$, where $C(\sigma) = \{c \in R: (cR + P)/P \text{ is essential in } R/P\}$, and T is a right ideal of R . But if $C = C(\sigma) = C(\tau)$ then as in (1.7) it follows that $R_\tau = C^{-1}R = RC^{-1}$; hence R_τ is left and right R -flat.

REMARK 1.10. Theorem (1.6) can be modified for rings R which are finitely generated as modules over a commutative Noetherian ring K in the following way: “ R has (M2) $\Leftrightarrow$ every ideal of R_P is reflexive”. Here $P = \{x \in K | xA = 0\}$ where A ranges over $\alpha - 1$ criticals, and R_P denotes localisation at the multiplicative set $K - P$. (P is prime because A is critical.) To prove this, it suffices to note that, by (1.8), R has (M2) $\Leftrightarrow$ for every essential I in R and every P as above, there is x in $K - P$ with $xI^{**} \subseteq I$. The result now follows easily. This leads us to ask the following question: “Can a classification theorem similar to (1.6) be obtained for fully bounded rings (or for all rings) by either modifying the localisations in a suitable way or by restricting them to only some height one localisations?”

§2

DEFINITIONS 2.1. (i) An R -module M is said to be $\alpha - 1$ unmixed if any nonzero finitely generated submodule of M has Krull dimension $\alpha - 1$.

(ii) We say R has the height one intersection property if $R = \bigcap R_\tau$ where the intersection is taken over all height one localisations τ .

LEMMA 2.2. *R has the height one intersection property if and only if Q/R is $\alpha - 1$ unmixed.*

PROOF. Let $0 \neq \bar{a} \in Q/R$. Suppose $|Ra + R/R| = \alpha - 1$. Then by (1.2) there exists a height one localisation τ such that $(Ra + R/R)_\tau \neq 0$; hence $a \notin R_\tau$. Thus if Q/R is $\alpha - 1$ unmixed then $R = \bigcap R_\tau$. Conversely assume that R has the height one intersection property. If $|Ra + R/R| < \alpha - 2$ then $R_\tau a \subseteq R_\tau$ for any τ of height one, and hence $a \in R$. It follows that $|Ra + R/R| = \alpha - 1$ and Q/R is $\alpha - 1$ unmixed.

LEMMA 2.3. *Let R have (M2) and the height one intersection property. Let $I \subseteq J$ be regular ideals. Then $I^* = J^*$ if and only if $|J/I| < \alpha - 2$.*

PROOF. If $I^* = J^*$ then $J/I \subseteq I^{**}/I$; hence $|J/I| < \alpha - 2$ by (M2). To show the converse first note that $|R/L| < \alpha - 2$, L a regular ideal $\Rightarrow L^* = R$. For, if $q \in L^* - R$ we get a nonzero homomorphism $R/L \rightarrow Q/R$; hence $|R/L| = \alpha - 1$ by (2.2). Now assume $|J/I| < \alpha - 2$ and let $J = J_0 \supseteq J_1 \supseteq \dots \supseteq J_n = I$ be a chain of ideals such that $J_i/J_{i+1} \simeq R/L_i$, for some ideals L_i , $i = 0, \dots, n - 1$. Each L_i is regular and $|R/L_i| < \alpha - 2$. Hence $L_i^* = R$, which implies that $\text{Ext}(R/L_i, R) = 0$. Therefore $\text{Ext}(J_i/J_{i+1}, R) = 0$ for all i and it follows that $\text{Ext}(J/I, R) = 0$, and hence $J^* = I^*$.

THEOREM 2.4. *The following statements are equivalent.*

- (i) R has (M2) and Q/R is $\alpha - 1$ unmixed.
- (ii) R has (M2) and the height one intersection property.
- (iii) For any regular ideals $I \subseteq J$, $I^* = J^*$ if and only if $|J/I| < \alpha - 2$.

- (iv) (a) *A proper regular ideal I is reflexive if and only if R/I is $\alpha - 1$ unmixed;*
 (b) *for regular ideals $I \subseteq J$, $I^* = J^*$ implies $\text{Ext}(J/I, R) = 0$;*
 (c) *if L is a regular ideal with $L^* = R$ then $|R/L| < \alpha - 2$.*

PROOF. By (2.2) and (2.3) we have (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). By assumption (c) holds, and to show (b) choose a chain of ideals $J = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n = I$ with $J_i/J_{i+1} \simeq R/L_i$ for $i = 0, \dots, n-1$. Then $I^* = J^* \Rightarrow |J/I| < \alpha - 2$ by (iii). Hence $|R/L_i| < \alpha - 2$, and so $L_i^* = R$ again by (iii). So $\text{Ext}(R/L_i, R) = 0$ and it follows that $\text{Ext}(J/I, R) = 0$. To see (a): let $I = I^{**} \subset K$ such that $|K/I| < \alpha - 2$. Then $I^* = K^*$, so $K = I$. Conversely if R/I is $\alpha - 1$ unmixed and $I \neq I^{**}$ then $|I^{**}/I| = \alpha - 1$, which by (iii) gives $I^* \neq I^{**}$.

(iv) $\Rightarrow$ (i). Let I be a regular ideal. Let $I \subseteq J \subseteq I^{**}$ such that $J/I \simeq R/L$. Then $I^* = J^*$ and L is a regular ideal. Hence $L^*/R \simeq \text{Ext}(R/L, R) \simeq \text{Ext}(J/I, R) = 0$, by (b). Thus by (c), $|R/L| < \alpha - 2$. Therefore $|J/I| < \alpha - 2$ and R has (M2). If now N is a finitely generated (nonzero) submodule of Q/R then $N \simeq R/Rc$ for some regular element c in R . Hence $|R/Rc| = \alpha - 1$ by (a).

Next we classify finitely generated reflexive modules over rings which satisfy the equivalent conditions of (2.4).

THEOREM 2.5. *Let R satisfy the equivalent conditions of (2.4). Then the following statements are equivalent on a finitely generated R -module M .*

- (i) *M is reflexive.*
 (ii) *There exists a finitely generated free module F containing M such that $\text{Hom}(R/I, F/M) = 0$ for any regular ideal I satisfying $I^* = R$.*
 (iii) *There exists a finitely generated free module F containing M such that either $F = M$ or F/M has no nonzero submodule of Krull dimension $< \alpha - 2$.*
 (iv) *M is torsionless and $\text{Ext}(R/I, M) = 0$ for any regular ideal I for which $I^* = R$.*

PROOF. (i) $\Rightarrow$ (ii). There exists an exact sequence $0 \rightarrow M \rightarrow F \rightarrow G$ with F, G finitely generated free. Hence for any regular ideal I (and in particular if $I^* = R$) we have $\text{Hom}(R/I, F/M) = 0$.

(ii) $\Leftrightarrow$ (iii). This is easily verified.

(ii) $\Rightarrow$ (iv). Certainly M is torsionless and if I is a regular ideal for which $I^* = R$ then the exact sequence $0 \rightarrow M \rightarrow F \rightarrow F/M \rightarrow 0$ yields

$$\text{Hom}(R/I, F/M) \rightarrow \text{Ext}(R/I, M) \rightarrow \text{Ext}(R/I, F)$$

in which both end terms vanish.

(iv) $\Rightarrow$ (ii). This follows from the fact that since M is torsionless there exists an exact sequence $0 \rightarrow M \rightarrow F \rightarrow F/M \rightarrow 0$ in which F is finitely generated free.

(ii) $\Rightarrow$ (i). Suppose that F as described in (ii) exists. We have an exact sequence $0 \rightarrow M \rightarrow M^{**} \rightarrow N \rightarrow 0$, and we want to show that $N = 0$. Since R has (M2) it follows that $|N| < \alpha - 2$. Suppose $N \neq 0$. Then there exists an ideal I such that $|R/I| < \alpha - 2$ and $\text{Hom}(R/I, N) \neq 0$. Thus I is regular with $I^* = R$. From the

above sequence we obtain

$$\text{Hom}(R/I, M^{**}) \rightarrow \text{Hom}(R/I, N) \rightarrow \text{Ext}(R/I, M)$$

exact in which the first term vanishes. We obtain a contradiction by showing that $\text{Ext}(R/I, M) = 0$. Now the exact sequence $0 \rightarrow M \rightarrow F \rightarrow F/M \rightarrow 0$ yields $\text{Hom}(R/I, F/M) \rightarrow \text{Ext}(R/I, M) \rightarrow \text{Ext}(R/I, F)$ exact. By (ii) the first term vanishes, and so does $\text{Ext}(R/I, F)$ since F is finitely generated free.

EXAMPLES 2.6. Among Noetherian prime rings satisfying the equivalent conditions of (2.4) are:

- (i) Commutative Noetherian integrally closed domains R with $0 < |R| < \infty$.
- (ii) Certain simple Ore extensions [4, 3.5].
- (iii) Any flat maximal K -order in a central simple algebra over the quotient field of K , where K is a commutative Noetherian integrally closed domain [3, 4.2].

§3

In this section, we study rings R that have the property that R_τ is hereditary for every height one localisation τ . We need:

DEFINITION 3.1. An element x of R is said to be critical if Rx is a critical R -module. When x is critical, $c(x)$ denotes the sum of critical ideals of R which have nonzero intersection with Rx .

LEMMA 3.2. Suppose $x \in R$ is critical and $x^2 \neq 0$. Then

- (i) $c(x)$ is critical.
- (ii) $c(x) \cap l(x) = 0$ where $l(x)$ is the left annihilator of x .
- (iii) $l(x) \oplus Rx$ and $l(x) \oplus c(x)$ are essential in R .
- (iv) If d is regular in R and $a \in R$ then $Ra \cap Rad \neq 0$.

PROOF. Since R is prime, $c(x)$ is nonsingular and (i) is clear. If $0 \neq c(x) \cap l(x)$, $|Rx^2| < |Rx|$. However Rx is critical; hence $x^2 = 0$, a contradiction. This shows (ii). Since $R/l(x) \simeq Rx$ is critical and $Rx \neq 0$, $l(x) \oplus Rx$ is essential in R ; hence (iii) follows. Finally for (iv), $Ra \cap Rad = 0$ implies that the sum $\sum_i Rad^i$ is direct, a contradiction.

THEOREM 3.3. Let R have (M2) and the height one intersection property. Suppose that all height one localisations over R are hereditary. Let $x \in R$ be critical, $x^2 \neq 0$. Then $R = c(x) \oplus l(x)$.

PROOF. Let τ be of height one, $S = R_\tau$ and $L =$ left annihilator of x in S . Since S is hereditary and since projective ideals are reflexive, (1.5) shows that $SI = I_\tau$ for any ideal I of R . For any ideal K of S , since $K/K \cap R$ is τ -torsion, $K = (K \cap R)_\tau = S(K \cap R)$. Hence $L = SI(x)$ and the sequence

$$0 \rightarrow L \rightarrow S \rightarrow Sx \rightarrow 0$$

is exact and splits, say by $x \rightarrow y$ for some y in S . Then $yx = x$ and as x, y have equal annihilators, $(1 - y)x = 0$ implies that $(1 - y)y = 0$, that is, $y = y^2$. Let $d = x + (y - 1)$. Then $x = yx = yd \in Sd$ and $y - 1 = (1 - y)d \in Sd$; hence $Sd \supseteq Sx + S(y - 1)$. For any ideal I of R , SI/I is τ -torsion; hence I is essential in SI . Therefore by (3.2(iii)), $Sx \cap L = 0$ and $Sx \oplus L$ is essential. But $L = S(y - 1)$, and Sd contains the essential ideal $Sx \oplus L$; hence d is regular in

S . Since $yd = x$, it follows from (3.2(iv)) that $Sx \cap Sy \neq 0$. Let $I = Sy \cap R$. Then $I \cap l(x) = 0$ and $I \cap Rx \neq 0$. Thus $I \subseteq c(x)$. Since $SI = Sy$, we get $S(l(x) \oplus c(x)) = S$. It follows from (1.2) that $|R/l(x) \oplus c(x)| < \alpha - 2$, and hence from (2.3) that $l(x)^{**} \oplus c(x)^{**} \simeq R$. This isomorphism is identity on $l(x)$ and on $c(x)$. Therefore $R = T \oplus J$ with $T \supseteq l(x)$, $J \supseteq c(x)$ and $|T/l(x)| < \alpha - 2$. It follows that $T = l(x)$ and that J is critical. From (3.1), $J = c(x)$ and the proof is complete.

REMARK 3.4. Let R be a Noetherian prime ring, $R \subseteq S_\beta \subseteq Q$ where the S_β are hereditary rings which are R -flat. Then if $R = \cap S_\beta$, the conclusion of (3.3) is valid for R . Observe that if $S_\beta I = S_\beta$ for all β , then $I^* \subseteq S_\beta$ for all β ; hence $I^* = R$. The rest of the proof of (3.3) holds unchanged.

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