THE SEQUENTIAL STABILITY INDEX AND CERTAIN SPACES OF AFFINE FUNCTIONS

F. JELLETT

ABSTRACT. A theorem of J. M. Anderson and J. E. Jayne on the existence of uniformly closed linear subspaces of the space $C^1(X)$ of bounded Baire-1 functions on the compact Hausdorff space X having specified sequential stability index $\alpha < \Omega$ is here adapted and extended to yield similar theorems for certain spaces of Baire-affine functions defined on a compact Choquet simplex.

Let $S_K = S_K(X)$, $K = \mathcal{R}$ or $\mathcal{C}$ be a set of bounded K -valued functions on a set X and let S_K^1 be the set of pointwise limits of bounded, pointwise convergent sequences in S_K .

For each ordinal α we inductively define

$$S_K^\alpha = \left[\bigcup_{\beta < \alpha} S_K^\beta \right]^1.$$

The (pointwise) sequential stability index of S_K , denoted by $i[S_K]$, is the least ordinal α such that $S_K^\alpha = S_K^{\alpha+1}$. This definition was given by Anderson and Jayne in [2], the main result of which established the existence of uniformly closed linear subspaces M_α of $C_K^1(X)$, where $C_K(X)$ is the space of continuous K -valued functions on the topological space X , with $i[M_\alpha] = \alpha$ for each ordinal $\alpha < \Omega$, provided X is a nondispersed compact Hausdorff space. Their methods hinged on results of Sarason [5], [6] which imply the existence of uniformly closed subspaces M_α of the Banach space $l_K^\infty(\mathcal{N})$ of all bounded K -valued functions on the natural numbers $\mathcal{N}$ with $i[M_\alpha] = \alpha$ for each $\alpha < \Omega$. See also Jayne [3, Theorem 10].

The purpose of this note is to give similar results for the space $A(K)$ of continuous real-valued affine functions defined on a compact Choquet simplex (see Alfsen [1] for all relevant definitions). As in [3, Theorem 10], we employ Sarason's results by seeking to embed $l^\infty = l_{\mathcal{N}}^\infty(\mathcal{N})$ isometrically in a space $A^\alpha(K)$ in such a way that a sequence $(f_n) \in l^\infty$ converges pointwise over $\mathcal{N}$ if and only if its image in $A^\alpha(K)$ converges pointwise over K . However, Jayne's methods are not applicable here and we have to resort to other means. $A_s(K)$ denotes the linear space of all affine functions $f = g - h$ on K with g, h bounded affine and upper semicontinuous (u.s.c.).

THEOREM. *Let K be an (infinite-dimensional) compact Choquet simplex. Then there exists a positive linear isometry $T: l^\infty \rightarrow A_s^1(K)$ such that whenever (f_n) is a pointwise convergent sequence in l^∞ , the sequence (Tf_n) converges pointwise over K .*

Received by the editors August 11, 1980 and, in revised form, November 3, 1980.

1980 *Mathematics Subject Classification.* Primary 46E99; Secondary 26A21.

© 1981 American Mathematical Society
0002-9939/81/0000-0408/\$01.75

PROOF. Let $\sigma = (x_i)$ be a sequence of distinct elements of the set $\partial_e K$ of extreme points of K .

Let $f \in l^\infty$ and define the functions g_n, h_n on K as follows:

$$\begin{aligned} g_n(x) &= f(i), \quad \text{if } x = x_i, 1 \leq i \leq n, \text{ and } f(i) > 0, \\ &= 0, \quad \text{otherwise.} \\ h_n(x) &= -f(i), \quad \text{if } x = x_i, 1 \leq i \leq n, \text{ and } f(i) < 0, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then g_n, h_n are convex and u.s.c., so their upper envelopes, $\hat{g}_n, \hat{h}_n$ [1, p. 4] are affine and u.s.c. [1, p. 89, Theorem II.3.8]. Let $f_n = \hat{g}_n - \hat{h}_n \in A_s(K)$, $n = 1, 2, \dots$

Define the map T as follows

$$(Tf)(x) = \tilde{f}(x) = \int_\sigma \tilde{f} d\mu_x$$

where, for each $x \in K$, μ_x is the unique maximal representing measure for x [1, p. 87] and, for $x = x_i \in \sigma$, $\tilde{f}(x_i)$ is defined to be equal to $f(i)$. It immediately follows that $\tilde{f}$ vanishes on $\partial_e K \setminus \sigma$ since the maximal representing measure for any point $x \in \partial_e K$ is the point-mass ε_x .

Now let $x \in K$. Then

$$f_n(x) = \hat{g}_n(x) - \hat{h}_n(x) = \int_K \hat{g}_n d\mu_x - \int_K \hat{h}_n d\mu_x,$$

since the barycenter formula $a(x) = \int_K a d\mu_x$ is valid for any bounded u.s.c. affine function a . Since $\int_K \hat{p} d\mu = \int_K p d\mu$ for any bounded u.s.c. function p and maximal measure μ [1, p. 35, Remark 1], we have

$$\int_K (\hat{g}_n - \hat{h}_n) d\mu_x = \int_K (g_n - h_n) d\mu_x = \int_\sigma (g_n - h_n) d\mu_x$$

as g_n and h_n vanish off σ . Moreover, since $g_n - h_n$ converges to $\tilde{f}$ pointwise over σ , it finally follows from the dominated convergence theorem that

$$f_n(x) \rightarrow \tilde{f}(x).$$

Hence $\tilde{f} \in A_s^1(K)$ for each $f \in l^\infty$ and the map $T: l^\infty \rightarrow A_s^1(K)$ is easily shown to have the required properties.

COROLLARY. Let K be an (infinite-dimensional) metrizable Choquet simplex. Then for each ordinal $\alpha < \Omega$ there exists a uniformly closed linear subspace N_α of $A^2(K)$ with $i[N_\alpha] = \alpha$.

PROOF. Let $T: l^\infty \rightarrow A_s^1(K)$ be the map furnished by the above theorem. Since K is metrizable, each semicontinuous function on K is of Baire class 1 and, indeed, by a theorem of Mokobodzki [1, p. 189] we have $A_s(K) \subseteq A^1(K)$ in this case.

Hence $A_s^1(K) \subseteq A^2(K)$ and the image $T(M_\alpha) = N_\alpha$ of the space M_α yielded by Sarason's theorem clearly satisfies $i[N_\alpha] = \alpha$, for each $\alpha < \Omega$.

REMARK 1. If $\partial_e K$ is σ -compact, then each $f \in C^1(\partial_e K)$ extends uniquely to an element $\tilde{f} \in A^1(K)$ (see e.g. Lacey and Morris [4, Lemma 2.1]). In this case $A^1(K)$ is isometrically order-isomorphic with $C^1(\partial_e K)$ and we have a result closer to

Theorem 10 of [3]: for each ordinal $\alpha < \Omega$ there exists a uniformly closed linear subspace N_α of $A^1(K)$ with $i[N_\alpha] = \alpha$. However, we do not see how to achieve this in general.

REMARK 2. It is an open question whether or not, for each ordinal $\alpha < \Omega$, there exists a linear subspace N_α of $A(K)$ with $i[N_\alpha] = \alpha$. The methods of the present paper are plainly inapplicable here for they would call for a sequence of distinct elements of $\partial_e K$ with no cluster point in the compact space K , which is impossible.

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, Springer-Verlag, Berlin, 1971.
2. J. M. Anderson and J. E. Jayne, *The sequential stability index of a function space*, *Mathematika* **20** (1973), 210–213.
3. J. E. Jayne, *Spaces of Baire functions* 1, *Ann. Inst. Fourier (Grenoble)* **4** (1974), 47–76.
4. H. E. Lacey and P. D. Morris, *On spaces of type $A(K)$ and their duals*, *Proc. Amer. Math. Soc.* **23** (1969), 151–157.
5. D. Sarason, *A remark on the weak-star topology of l^∞* , *Studia Math.* **30** (1968), 355–359.
6. ———, *On the order of a simply-connected domain*, *Michigan Math. J.* **15** (1968), 129–133.

DEPARTMENT OF MATHEMATICS, CHELSEA COLLEGE, UNIVERSITY OF LONDON, LONDON, ENGLAND

Current address: Department of Mathematics, Auburn University, Auburn, Alabama 36849