MORE ON CONVERGENCE IN UNITARY MATRIX SPACES

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ABSTRACT. Let E be a symmetric sequence space satisfying the Radon-Riesz Property

$$\{\|x_n\| \rightarrow \|x\| \text{ and } x_n \rightarrow x \text{ weakly}\} \Rightarrow \|x_n - x\| \rightarrow 0,$$

then the same is true for the associated unitary matrix space C_E .

Let E be a symmetric sequence space, i.e., a Banach space of sequences so that the standard unit vectors $\{e_n\}_{n=1}^\infty$ (defined by $e_n(j) = \delta_{n,j}$) form a 1-symmetric, normalized basis of E . The *unitary matrix space* C_E associated with E is the Banach space of all compact operators x on l_2 for which $s(x) \in E$, normed by

$$\|x\|_{C_E} = \|s(x)\|_E.$$

Here $s(x) = (s_n(x))$ is the sequence of s -numbers of x , i.e., the eigenvalues of $(x^*x)^{1/2}$, arranged in a nonincreasing ordering, counting multiplicity.

In the recent paper [1] we reduce the study of certain properties of C_E to the study of the analogous properties of E . One consequence is the following result which characterizes convergence in C_E in terms of convergence in E (see [2]): *If $\{x_n\}$ and x are elements of C_E , then $\|x_n - x\|_{C_E} \rightarrow 0$ if and only if $\|s(x_n) - s(x)\|_E \rightarrow 0$ and $x_n \rightarrow x$ weakly.*

In this note we apply [1] to study some related convergence property, and show that it extends from E to C_E .

DEFINITION. A Banach space X is said to have the *Radon-Riesz Property* (RRP, in short) if $\|x_n\| \rightarrow \|x\|$ and $x_n \rightarrow x$ weakly imply $\|x_n - x\| \rightarrow 0$ for all $\{x_n\}$ and x in X .

THEOREM I. *Let E be a symmetric sequence space. Then E has the RRP if and only if C_E has the RRP.*

This theorem extends the known result for the case $E = l_p$, $1 < p < \infty$, (see [3]; or apply the uniform convexity of the spaces $C_p = C_l$, [4]), and answers affirmatively a question of B. Simon.

If C_E has the RRP, then the same is true for E which is isometric to the subspace of C_E consisting of all diagonal matrices. So the point in the theorem is that the RRP extends from E to C_E .

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Let us start with two propositions.

PROPOSITION 1. *Assume that E satisfies the RRP. Let $x \in E$, $\|x\| = 1$ and let $y \in E^*$ be a supporting functional for x : $\|y\| = 1 = \langle x, y \rangle$. Then y does not vanish on the support of x , i.e., $x(i) \neq 0$ implies $y(i) \neq 0$.*

PROOF. The assertion is clear if $E = l_1$. If $E \neq l_1$ then $e_n \rightarrow 0$ weakly. If $x(i_0) \neq 0$ but $y(i_0) = 0$, we get a contradiction as follows. Let $u = \sum_{i=1, i \neq i_0}^{\infty} x(i)e_i$. Then,

$$1 = \|x\| \geq \|u\| \geq |\langle u, y \rangle| = |\langle x, y \rangle| = 1.$$

So, $\|u\| = 1$. Define

$$x_n = \sum_{\substack{i=1 \\ i \neq i_0}}^n x(i)e_i + x(i_0)e_{n+1}.$$

Clearly, $\|x_n\| \rightarrow \|x\| = 1 = \|u\|$. Also $\sum_{i=1, i \neq i_0}^n x(i)e_i \rightarrow u$ in norm. So $x_n \rightarrow u$ weakly. But $\|x_n - u\| \rightarrow |x(i_0)| > 0$, contradicting the RRP. $\square$

PROPOSITION 2. *Assume that E has the RRP. Let $x \in E$ has nonnegative coordinates and let $\lambda > 0$. Then for all i ,*

$$\|x + \lambda e_i\| > \|x\|.$$

PROOF. Again, this is clear for $E = l_1$. Assume therefore that $E \neq l_1$. Clearly, $\|x + \lambda e_i\| \geq \|x\|$ for all i . If for some i_0 $\|x + \lambda e_{i_0}\| = \|x\|$, we get a contradiction as follows. If $x(i_0) > 0$, let $y \in E^*$ be a normalized element which supports x : $\langle x, y \rangle = \|x\|$. Clearly, we can assume that the coordinates of y are nonnegative. By Proposition 1, $y(i_0) > 0$. So,

$$\|x\| = \|x + \lambda e_{i_0}\| \geq |\langle x + \lambda e_{i_0}, y \rangle| = \|x\| + \lambda y(i_0) > \|x\|,$$

a contradiction. If $x(i_0) = 0$, define

$$x_n = \sum_{i=1}^n x(i)e_i + \lambda e_{n+1}.$$

Then $\|x_n\| = \|\sum_{i=1}^n x(i)e_i + \lambda e_{i_0}\| \rightarrow \|x + \lambda e_{i_0}\| = \|x\|$, and $x_n \rightarrow x$ weakly. But $\|x_n - x\| \rightarrow \lambda > 0$, contradicting the RRP. $\square$

PROOF OF THEOREM I. Let $\{x_n\}$ and x be elements of C_E satisfying

$$\|x_n\| \rightarrow \|x\|, \quad x_n \rightarrow x \text{ weakly.}$$

If $\|x_n - x\| \not\rightarrow 0$, then for some $\varepsilon > 0$ and some subsequence $y_k = x_{n_k} - x$ we have

$$\|y_k\| > \varepsilon \quad \text{for all } k.$$

Choose a matrix representation in which x is diagonal (with the s -numbers on the diagonal). Since $y_k \rightarrow 0$ weakly, we can assume (by passing to a subsequence if necessary and by a standard perturbation argument) that $\{y_k\}$ is a shell-block sequence, i.e., that for some increasing sequence $\{m_k\}$ we have $y_k(i, j) = 0$ if $\max\{i, j\} \leq m_k$ or $\max\{i, j\} > m_{k+1}$. By Theorem 2.4 of [1] we can assume, furthermore, that there exists a tensor product representation and a further subsequence, still denoted by $\{y_k\}$ for simplicity, so that

$$y_k = \sum_{i=1}^{k-1} (a_i \otimes e_{k,i} + b_i \otimes e_{i,k}) + c_k \otimes e_{k,k}$$

and

$$x = \sum_{k=1}^{\infty} z_k \otimes e_{k,k}$$

with z_k diagonal matrices.

CLAIM. $a_i = b_i = 0$ for all i .

Indeed, if $a_{i_0}(\nu, \mu) = \alpha \neq 0$ for some i_0, ν and μ , we define $\tilde{z}_{i_0}$ by

$$\tilde{z}_{i_0}(\mu, \mu) = (|z_{i_0}(\mu, \mu)|^2 + |\alpha|^2)^{1/2}$$

and

$$\tilde{z}_{i_0}(s, t) = z_{i_0}(s, t) \quad \text{otherwise.}$$

Let

$$\tilde{x} = \sum_{\substack{i=1 \\ i \neq i_0}}^{\infty} z_i \otimes e_{i,i} + \tilde{z}_{i_0} \otimes e_{i_0,i_0}.$$

By Proposition 2, $\|x\| < \|\tilde{x}\|$. On the other hand, if we define for $i_0 < m$ a projection P_m by

$$P_m \left(\sum_{k,l} u_{k,l} \otimes e_{k,l} \right) = \sum_{k=1}^m u_{k,k} \otimes e_{k,k} + u_{m+1,i_0}(\nu, \mu) e_{\nu,\mu} \otimes e_{m+1,i_0}$$

then P_m is contractive and thus,

$$\begin{aligned} \|x_{n_m}\| &> \|P_m x_{n_m}\| = \left\| \sum_{k=1}^m z_k \otimes e_{k,k} + \alpha e_{\nu,\mu} \otimes e_{m+1,i_0} \right\| \\ &= \left\| \sum_{\substack{k=1 \\ k \neq i_0}}^m z_k \otimes e_{k,k} + \tilde{z}_{i_0} \otimes e_{i_0,i_0} \right\| \rightarrow \|\tilde{x}\|. \end{aligned}$$

This is a contradiction, since $\|x_{n_m}\| \rightarrow \|x\| < \|\tilde{x}\|$. This shows that $\alpha = 0$. Thus $a_i = 0$ for all i . The proof that $b_i = 0$ for all i is the same. This concludes the proof of the claim.

It follows that $y_k = c_k \otimes e_{k,k}$ for all k . Let d_k be a diagonal matrix whose diagonal entries are the s -numbers of c_k , and define for $m = 1, 2, \dots$

$$u_m = \sum_{k=1}^{m-1} z_k \otimes e_{k,k} + d_m \otimes e_{m,m}.$$

u_m is a diagonal matrix, and thus we regard it as an element of E . Now,

$$\begin{aligned} \left\| \sum_{k=1}^{m-1} z_k \otimes e_{k,k} \right\| &< \|u_m\| = \left\| \sum_{k=1}^{m-1} z_k \otimes e_{k,k} + y_m \right\| \\ &= \left\| x_{n_m} - \sum_{k=m}^{\infty} z_k \otimes e_{k,k} \right\|. \end{aligned}$$

So,

$$(*) \quad \|u_m\| \rightarrow \|x\|.$$

Also, let $\beta = \sum_{k=1}^{\infty} \beta_k \otimes e_{k,k}$ be any diagonal matrix in $(C_E)^* = C_{E^*}$ (i.e., a member of E^*). For all k let v_k, w_k be unitary matrices so that $v_k c_k w_k = d_k$. Then

$$\begin{aligned} \langle d_k \otimes e_{k,k}, \beta \rangle &= \langle d_k, \beta_k \rangle = \text{trace}(v_k c_k w_k \beta_k^*) \\ &= \text{trace}(c_k (v_k^* \beta_k w_k^*)^*) = \langle c_k, v_k^* d_k w_k^* \rangle. \end{aligned}$$

Let $v = \sum_{k=1}^{\infty} v_k \otimes e_{k,k}$, $w = \sum_{k=1}^{\infty} w_k \otimes e_{k,k}$. Then v, w are unitary and

$$\langle d_k \otimes e_{k,k}, \beta \rangle = \langle y_k, v^* \beta w^* \rangle \rightarrow 0.$$

So, clearly,

$$(**) \quad u_m \rightarrow x \text{ weakly.}$$

But

$$\begin{aligned} \|u_m - x\| &\geq \|d_m \otimes e_{m,m}\| - \left\| \sum_{k=m}^{\infty} z_k \otimes e_{k,k} \right\| \\ &\geq \|y_m\| - \varepsilon/2 > \varepsilon - \varepsilon/2 = \varepsilon/2 \end{aligned}$$

if m is large enough. This, together with (*) and (**) contradicts the assumption that E has the RRP, and thus concludes the proof of Theorem I. $\square$

In exactly the same way one can prove the following result.

THEOREM II. *Let E be a symmetric sequence space. The following two statements are equivalent.*

(1) *If $\{x_n\}$ and x are elements of E satisfying*

(†) $\|x_n\|_E \rightarrow \|x\|_E$ and $x_n(i) \rightarrow x(i)$ for all i , then $\|x_n - x\|_E \rightarrow 0$.

(2) *If $\{x_n\}$ and x are elements of C_E satisfying*

(††) $\|x_n\|_{C_E} \rightarrow \|x\|_{C_E}$ and $x_n \rightarrow x$ in the weak operator topology, then $\|x_n - x\|_{C_E} \rightarrow 0$.

Appendix. We decide to add to the paper a direct proof of Theorems I and II in the important special case of the trace class $C_1 = C_{l_1}$, which does not depend on the elaborate blocking technique of [1]. The proof presented here is an improvement, due to B. Simon, of our original proof. We would like to thank Professor Simon for his comments and interest.

It is easy to see that (1) of Theorem II is satisfied in case $E = l_1$. So Theorems I and II in the case $E = l_1$ are reduced to the following

THEOREM. *Let $\{x_n\}_{n=1}^{\infty} \subseteq C_1$ and $x \in C_1$ be so that*

(i) $\|x_n\|_1 \rightarrow \|x\|_1$;

(ii) $x_n \rightarrow x$ in the weak operator topology. Then $\|x_n - x\|_1 \rightarrow 0$.

Here, $\|\cdot\|_1$ is, of course, the trace-norm, i.e.,

$$\|x\|_1 = \text{trace}(x^* x)^{1/2} = \sum_{j=1}^{\infty} s_j(x).$$

The proof of the theorem is based on the following inequality, which is of independent interest.

PROPOSITION. Let $x \in C_1$, let p be an orthogonal projection in l_2 and let $q = I - p$. Then

$$(iii) \|x\|_1^2 \geq \|pxp\|_1^2 + \|pxq\|_1^2 + \|qxp\|_1^2 + \|qxq\|_1^2.$$

SKETCH OF PROOF. For every $y \in B(l_2)$ we have ($\|\cdot\|_\infty$ is the operator norm)

$$\begin{aligned} \|y\|_\infty &= \sup\{\|y\xi\|; \|\xi\| \leq 1\} = \sup\{(\|py\xi\|^2 + \|qy\xi\|^2)^{1/2}; \|\xi\| \leq 1\} \\ &\leq (\|py\|_\infty^2 + \|qy\|_\infty^2)^{1/2}. \end{aligned}$$

From this one gets

$$\|y\|_\infty \leq (\|pyy\|_\infty^2 + \|pyq\|_\infty^2 + \|qyp\|_\infty^2 + \|qyq\|_\infty^2)^{1/2}$$

and from this (iii) follows by standard duality arguments. $\square$

PROOF OF THE THEOREM. By (ii), there exists an increasing sequence $\{p_n\}_{n=1}^\infty$ of finite rank orthogonal projections in l_2 so that $p_n \uparrow I$ and

$$(iv) \|p_n x_n p_n - x\|_1 \rightarrow 0.$$

Let $q_n = I - p_n$. Then, by (iii)

$$\begin{aligned} \|x_n\|_1^2 &\geq \|p_n x_n p_n\|_1^2 + \|p_n x_n q_n\|_1^2 + \|q_n x_n p_n\|_1^2 + \|q_n x_n q_n\|_1^2 \\ &\geq \|p_n x_n p_n\|_1^2 + (\|p_n x_n q_n\|_1 + \|q_n x_n p_n\|_1 + \|q_n x_n q_n\|_1)/3. \end{aligned}$$

So,

$$\begin{aligned} \|x_n - x\|_1 &\leq \|p_n x_n p_n - x\|_1 + \|p_n x_n q_n\|_1 + \|q_n x_n p_n\|_1 + \|q_n x_n q_n\|_1 \\ &\leq \|p_n x_n p_n - x\|_1 + 3(\|x_n\|_1^2 - \|p_n x_n p_n\|_1^2). \end{aligned}$$

By (i) and (iv), the expression on the right-hand side of the last inequality tends to zero. This completes the proof of the theorem. $\square$

REFERENCES

1. J. Arazy, *Basic sequences, embeddings and the uniqueness of the symmetric structure in unitary matrix spaces*, J. Funct. Anal. (to appear).
2. ———, *On the geometry of the unit ball of unitary matrix spaces*, Integral Equations Operator Theory (to appear).
3. H. R. Grümmer, *Two theorems about C_p* , Rep. Math. Phys. **4** (1973), 211–215.
4. N. Tomczak-Jaegermann, *The moduli of smoothness and convexity and Rademacher averages of trace classes S_p ($1 < p < \infty$)*, Studia Math. **50** (1974), 163–182.

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