ON A THEOREM OF STANOJEVIĆ

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ABSTRACT. A new direct proof of a theorem of Stanojević is given. Consequently it is proven that the Fomin class $\mathcal{F}_p$ ($1 < p < 2$) is a subclass of the class $\mathcal{BV} \cap \mathcal{C}$, where $\mathcal{C}$ is the Garrett-Stanojević class and $\mathcal{BV}$ is the class of null sequences of bounded variation. This also provides a new direct proof of Fomin's theorem.

Concerning the problem of necessary and sufficient conditions for the L^1 -convergence of the cosine series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

with coefficients in the class of null sequences of bounded variation ($\{a_n\} \in \mathcal{BV}$), a theorem in [1] can be reformulated as

THEOREM (GARRETT AND STANOJEVIĆ [1]). *Let $\{a_n\} \in \mathcal{BV} \cap \mathcal{C}$. Then $f \in L^1(0, \pi)$ and*

$$\|S_n - f\| = o(1) \quad (n \rightarrow \infty) \text{ if and only if } a_n \lg n = o(1) \quad (n \rightarrow \infty). \quad (Y)$$

The class $\mathcal{C}$ was defined in [2] as follows:

$\{a_n\} \in \mathcal{C}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^\delta \left| \sum_{k=n}^{\infty} \Delta a_k D_k(x) \right| dx < \varepsilon, \quad \text{for all } n,$$

where $D_n(x) = \frac{1}{2} + \sum_{k=1}^n \cos kx$ is the Dirichlet kernel.

Fomin [3] proved that if, for some $1 < p < 2$,

$$\sum_{n=1}^{\infty} \left(\frac{\sum_{k=n}^{\infty} |\Delta a_k|^p}{n} \right)^{1/p}, \quad (1)$$

converges, then $f \in L^1(0, \pi)$ and (Y) holds. The class of all null sequences satisfying Fomin's condition is denoted $\mathcal{F}_p$.

In [2], Stanojević has shown that $\mathcal{F}_p \subset \mathcal{BV} \cap \mathcal{C}$. However the proof of Stanojević involves certain unnecessary consequences of $\{a_n\} \in \mathcal{F}_p$. In this paper a direct proof of this theorem is given which does not involve these cumbersome consequences.

THEOREM. $\mathcal{F}_p \subseteq \mathcal{BV} \cap \mathcal{C}$.

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PROOF. It suffices to show [1] that

$$C_n = \int_0^\pi \left| \sum_{k=n}^\infty \Delta a_k D_k(x) \right| dx = o(1) \quad (n \rightarrow \infty).$$

For each n , let k_n be the least natural number such that $n < 2^{k_n} - 1$. Then C_n can be majorized by

$$\begin{aligned} C_n &\leq \int_0^\pi \left| \sum_{j=n}^{2^{k_n}-1} \Delta a_j D_j(x) \right| dx + \sum_{l=k_n}^\infty \int_0^\pi \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx \\ &= I_1 + I_2. \end{aligned}$$

The second term is written as follows:

$$\begin{aligned} I_2 &= \sum_{l=k_n}^\infty \left\{ \int_0^{1/2^{l+1}} + \int_{1/2^{l+1}}^\pi \right\} \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j D_j(x) \right| dx \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

For the first term, the uniform estimate, $|D_n(x)| < n + \frac{1}{2}$, is applied, i.e.,

$$\Sigma_1 < \sum_{l=k_n}^\infty \frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j| \left(j + \frac{1}{2} \right) < \sum_{l=2^{k_n}}^\infty |\Delta a_j|.$$

The second sum is estimated by applying the Hölder inequality ($1/p + 1/q = 1$) followed by the Hausdorff-Young inequality

$$\begin{aligned} \Sigma_2 &\leq \sum_{l=k_n}^\infty \left(\int_{1/2^{l+1}}^\pi \frac{dx}{(2 \sin x/2)^p} \right)^{1/p} \left[\int_{1/2^{l+1}}^\pi \left| \sum_{j=2^l}^{2^{l+1}-1} \Delta a_j \sin \left(j + \frac{1}{2} \right) x \right|^q dx \right]^{1/q} \\ &\leq \frac{\pi}{2(p-1)} \sum_{l=k_n}^\infty 2^{(l+1)(1-1/p)} \left(\sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p} \\ &= \frac{\pi}{2(p-1)} \sum_{l=k_n}^\infty 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p}. \end{aligned}$$

The same method applied to I_1 yields the estimate

$$\begin{aligned} I_1 &\leq \sum_{l=n}^{2^{k_n}-1} |\Delta a_l| + 2^{k_n} \left(\frac{1}{2^{k_n}} \sum_{j=n}^{2^{k_n}-1} |\Delta a_j|^p \right)^{1/p} \\ &\leq \sum_{l=2^{k_n}-1}^{2^{k_n}-1} |\Delta a_l| + \frac{\pi}{2(p-1)} 2^{k_n} \left(\frac{1}{2^{k_n}} \sum_{j=2^{k_n}-1}^{2^{k_n}-1} |\Delta a_j|^p \right)^{1/p}. \end{aligned}$$

Combining the estimates, we have

$$C_n \leq \sum_{l=2^{k_n}-1}^\infty |\Delta a_l| + \frac{\pi}{2(p-1)} \sum_{l=k_n-1}^\infty 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p}.$$

The proof will be concluded by showing that (1) is equiconvergent with

$$\sum_{l=0}^{\infty} 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p}. \quad (2)$$

Applying the Cauchy condensation test, the series (1) is equiconvergent with

$$\sum_{l=0}^{\infty} 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{\infty} |\Delta a_j|^p \right)^{1/p}.$$

Thus, convergence of (1) implies convergence of (2). Conversely, for each $n > 1$,

$$\begin{aligned} \sum_{l=1}^n 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p} &= \sum_{l=1}^n 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{\infty} |\Delta a_j|^p - \frac{1}{2^{l+1}} \sum_{j=2^{l+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} \\ &> \sum_{l=1}^n 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{\infty} |\Delta a_j|^p \right)^{1/p} - \sum_{l=1}^n 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^{l+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} \\ &= (2^{1-1/p}) \sum_{l=0}^{n-1} 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^{l+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} - \sum_{l=1}^n 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^{l+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} \\ &= (2^{1-1/p} - 1) \sum_{l=1}^{n-1} 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^{l+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} \\ &\quad + 4^{1-1/p} \left(\sum_{j=2}^{\infty} |\Delta a_j|^p \right)^{1/p} - 2^{n+1} \left(\frac{1}{2^{n+1}} \sum_{j=2^{n+1}}^{\infty} |\Delta a_j|^p \right)^{1/p}. \end{aligned}$$

The convergence of (1) will follow from the condensation test if it is shown that the convergence of (2) implies

$$2^{n+1} \left(\frac{1}{2^{n+1}} \sum_{j=2^{n+1}}^{\infty} |\Delta a_j|^p \right)^{1/p} = o(1) \quad (n \rightarrow \infty).$$

To show this, we have for each $N > n + 1$,

$$\begin{aligned} 2^{n+1} \left(\frac{1}{2^{n+1}} \sum_{j=2^{n+1}}^{2^{N+1}-1} |\Delta a_j|^p \right)^{1/p} &= 2^{(n+1)(1-1/p)} \left(\sum_{l=n+1}^N \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p} \\ &\leq 2^{(n+1)(1-1/p)} \sum_{l=n+1}^N \left(\sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p} \\ &\leq \sum_{l=n+1}^N 2^{l+1} \left(\frac{1}{2^{l+1}} \sum_{j=2^l}^{2^{l+1}-1} |\Delta a_j|^p \right)^{1/p}. \end{aligned}$$

Letting $N \rightarrow +\infty$, the proof of the theorem follows.

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