ON MAHLER'S MEASURE OF A POLYNOMIAL

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ABSTRACT. Let P be a polynomial with complex coefficients. We denote by $M(P)$ (resp. $\|P\|$) the Mahler measure of P (resp. the maximum modulus of P on the disk $|z| < 1$). We prove here that $M(P) = \inf \|PQ\|$ where the infimum is taken over all polynomials Q with complex coefficients satisfying $Q(0) = 1$.

We denote by $\mathbb{C}[z]$ the ring of polynomials in z with complex coefficients. Given $P \in \mathbb{C}[z]$,

$$P(z) = a \prod_{j=1}^n (z - \alpha_j), \quad a \neq 0,$$

we define $M(P)$, the Mahler measure of P , by

$$M(P) = \exp\left(\int_0^1 \text{Log}|P(e^{2i\pi\theta})| d\theta\right).$$

By means of Jensen's formula, it can be shown (see Mahler [2]) that

$$M(P) = |a| \prod_{j=1}^n \max\{1, |\alpha_j|\}.$$

We further put $\|P\| = \sup_{|z|=1} |P(z)|$ and, for any real number $\mu > 0$,

$$\|P\|_\mu = \left(\int_0^1 |P(e^{2i\pi\theta})|^\mu d\theta\right)^{1/\mu}.$$

It is well known that $\mu \mapsto \|P\|_\mu$ is a nondecreasing function of μ which satisfies $\lim_{\mu \rightarrow 0} \|P\|_\mu = M(P)$ and $\lim_{\mu \rightarrow +\infty} \|P\|_\mu = \|P\|$.

In one of his papers, Lawton [1], has shown that

$$M(P) = \inf_{\substack{Q \in \mathbb{C}[z] \\ Q(0)=1}} \|PQ\|_2.$$

Here, we prove the following

THEOREM. For any polynomial $P \neq 0$, we have $M(P) = \inf_{Q \in \mathbb{C}[z]; Q(0)=1} \|PQ\|$. From this result, we easily deduce

$$M(P) = \inf_{\substack{Q \in \mathbb{C}[z] \\ Q(0)=1}} \|PQ\|_\mu$$

for every $\mu > 0$.

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PROOF OF THE THEOREM. If Q is a polynomial such that $Q(0) = 1$, then $M(Q) > 1$. Since $M(P) \leq M(P) \cdot M(Q) = M(PQ) \leq \|PQ\|$, we thus obtain

$$M(P) \leq \inf_{\substack{Q \in \mathbb{C}[z] \\ Q(0)=1}} \|PQ\|.$$

It will now be proved that, given $\varepsilon > 0$, we can find a polynomial Q such that $Q(0) = 1$ and $\|PQ\| \leq (1 + \varepsilon)M(P)$. To this end, we write

$$P(z) = a \prod_{j=1}^p (z - \alpha_j) \cdot \prod_{k=1}^q (z - \beta_k)$$

where $|\alpha_j| > 1$ ($j = 1, \dots, p$) and $|\beta_k| < 1$ ($k = 1, \dots, q$). Put $\delta = (1 + \varepsilon)^{1/p+q} - 1$, $r = 1 + \delta/2$, and let N be an integer such that $r^N > 2/\delta$. We further put

$$S(z) = \frac{(1 - z)}{r^N} \left(\frac{z^{N+1} - r^{N+1}}{z - r} \right).$$

The polynomial S satisfies $S(0) = 1$, $S(1) = 0$ and

$$\|S\| \leq \frac{(1 + r^{N+1})}{r^N} \cdot \sup_{|z|=1} \left| \frac{1 - z}{z - r} \right| \leq \frac{(1 + r^{N+1})}{r^N} \cdot \frac{2}{1 + r} < \frac{1}{r^N} + r.$$

We then consider the polynomial

$$Q(z) = \left((-1)^p \prod_{j=1}^p \alpha_j \right) \left(\prod_{j=1}^p \left\{ S\left(\frac{z}{\alpha_j}\right) (z - \alpha_j)^{-1} \right\} \right) \cdot \left(\prod_{k=1}^q \left\{ S(\bar{\beta}_k z) (1 - \bar{\beta}_k z)^{-1} \right\} \right).$$

It is easy to see that $Q(0) = 1$ and

$$|P(z)Q(z)| = M(P) \cdot \left(\prod_{j=1}^p \left| S\left(\frac{z}{\alpha_j}\right) \right| \right) \cdot \left(\prod_{k=1}^q |S(\bar{\beta}_k z)| \right) \left(\prod_{k=1}^q \left| \frac{z - \beta_k}{1 - \bar{\beta}_k z} \right| \right).$$

Since $|\alpha_j| \geq 1$ ($j = 1, \dots, p$) and $|\beta_k| < 1$ ($k = 1, \dots, q$), it follows from the maximum principle that

$$\max_{|z|=1} \left(\prod_{j=1}^p \left| S\left(\frac{z}{\alpha_j}\right) \right| \right) \left(\prod_{k=1}^q |S(\bar{\beta}_k z)| \right) \leq \|S\|^{p+q} \leq 1 + \varepsilon.$$

On the other hand $|(z - \beta_k)/(1 - \bar{\beta}_k z)| = 1$ for $|z| = 1$ ($k = 1, \dots, q$), so that

$$\max_{|z|=1} \prod_{k=1}^q \left| \frac{z - \beta_k}{1 - \bar{\beta}_k z} \right| = 1.$$

Hence, we get $\|PQ\| \leq (1 + \varepsilon)M(P)$, which complete the proof.

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