THE FUGLEDE COMMUTATIVITY THEOREM MODULO OPERATOR IDEALS

GARY WEISS

ABSTRACT. Let H denote a separable, infinite-dimensional complex Hilbert space. A two-sided ideal I of operators on H possesses the generalized Fuglede property (GFP) if, for every normal operator N and every $X \in L(H)$, $NX - XN \in I$ implies $N^*X - XN^* \in I$. Fuglede's Theorem says that $I = \{0\}$ has the GFP. It is known that the class of compact operators and the class of Hilbert-Schmidt operators have the GFP.

We prove that the class of finite rank operators and the Schatten p -classes for $0 < p < 1$ fail to have the GFP. The operator we use as an example in the case of the Schatten p -classes is multiplication by $z + w$ on L^2 of the torus.

Introduction. Let H denote a separable, infinite-dimensional complex Hilbert space. Let $L(H) \supset K(H) \supset C_p \supset F(H)$ ($0 < p < \infty$) denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten p -class, and the class of finite rank operators on H . All operators herein are assumed to be linear and bounded. Let $\|\cdot\|_p$ denote the C_p -norm.

Let I be any two-sided ideal in $L(H)$ (every ideal herein is assumed to be two-sided). It is well known that if $I \neq \{0\}$ or $L(H)$, then $K(H) \supset I \supset F(H)$.

DEFINITION. The ideal I is said to possess the *generalized Fuglede property* (GFP) if, for every normal operator N and every bounded operator X , we have $NX - XN \in I$ implies $N^*X - XN^* \in I$ (i.e., $NX = XN$ modulo I implies $N^*X = XN^*$ modulo I).

Fuglede's Theorem essentially states that $I = \{0\}$ has the GFP [1].

There is a connection between the GFP for C_1 (which is not known to hold true) and a possible generalization of the trace result [5, Question 3]. Namely, if N is a normal operator and X is a compact operator such that $NX - XN \in C_1$, must $\text{trace}(NX - XN) = 0$?

If C_1 possessed the GFP, then the answer to this question would be yes. The proof is the same as the proof for the well-known fact that if S is a selfadjoint operator and X is a compact operator then $SX - XS \in C_1$ implies $\text{trace}(SX - XS) = 0$ (cf. [8, p. 279, Lemma 1.3]). Say $T = NX - XN \in C_1$. If C_1 possessed the GFP, then $S = N^*X - XN^* \in C_1$. Hence $-S^* = NX^* - X^*N \in C_1$ and $T - S^* = N(X + X^*) - (X + X^*)N$ and $T + S^* = N(X - X^*) - (X - X^*)N$. But $X + X^*$ and $X - X^*$ are scalar multiples of compact, selfadjoint

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operators. By the spectral theorem they are diagonalizable. By a computation the above commutators must have trace 0. Therefore

$$\text{trace}(T - S^*) = \text{trace}(T + S^*) = 0.$$

Hence $\text{trace } T = 0$.

With these remarks, we pose the following questions:

1. Does C_1 possess the GFP?
2. If N is normal and X is compact, does $NX - XN \in C_1$ imply $\text{trace}(NX - XN) = 0$?
3. Are there any ideals in $L(H)$ other than $\{0\}$, C_2 and $K(H)$ that possess the GFP?

Some positive results. Let $|T|$ denote $\sqrt{T^*T}$.

The first interesting observation is that if I is any ideal and N is normal and $X \in L(H)$, then $|NX| = |N^*X|$. This is a trivial computation. In fact, if N is hyponormal, then $|NX| \geq |N^*X|$. From this we get N hyponormal implies $NX \in I \Rightarrow N^*X \in I$. Furthermore if N is normal and I is a normed ideal where $\|T\|_I = \||T|\|_I$ for all $T \in I$, then $\|NX\|_I = \|N^*X\|_I$. The same holds for XN and XN^* .

The compacts $K(H)$ have the GFP. This is clear by applying Fuglede's Theorem to the Calkin algebra.

The Class C_2 has the GFP [6, Theorem 1]. (For another proof use Voiculescu's Theorem [3].)

The Putnam generalization of Fuglede's Theorem [2] applies to the GFP. That is any ideal I has the GFP if and only if for every two normal operators N_1, N_2 and $X \in L(H)$, $N_1X - XN_2 \in I$ implies $N_1X - XN_2 \in I$. This is proved in an analogous way to the well-known proof of Putnam's generalization [2] using a 2×2 matrix trick due to S. K. Berberian [7].

The counterexamples.

THEOREM 1. *The class of finite rank operators $F(H)$ does not possess the GFP.*

PROOF. By the previous remark, it suffices to produce two trace class diagonal operators D_1, D_2 and a Hilbert-Schmidt operator X such that the rank of $D_1X - XD_2$ is 1, but the rank of $D_1^*X - XD_2^*$ is infinite.

Let D_1 denote the diagonal matrix with diagonal entries $1/2, 1/4, 1/8, \dots$. Clearly, $D_1 \in C_1$. Formally let D_2 denote the diagonal matrix with diagonal entries $z_1, z_2, \dots$ and $X_1 = (x_{ij})$ where $x_{ij} = 4^{-(i+j)}(2^{-i} - z_j)^{-1}$. We shall choose the sequence (z_n) inductively to satisfy several conditions. First of all, we want to be sure that $|2^{-i} - z_j|$ is large enough to insure that $X_1 \in C_2$, that is, x_{ij} is square summable. Secondly, we need to be sure that $|z_n|$ is small enough to insure that $D_2 \in C_1$.

In the first place, if we can choose (z_n) to be in the closed, left half-plane, then $|2^{-i} - z_j| \geq 2^{-i}$, and so $|x_{ij}| \leq 4^{-(i+j)}2^i = 2^{-i}4^{-j}$ which is square summable.

In the second place, we can insure that $D_2 \in C_1$ if we can choose (z_n) such that $|z_n| \leq 2^{-n}$.

In regard to the rank requirements on $D_1X_1 - X_1D_2$ and $D_1^*X_1 - X_1D_2^*$, note that

$$(D_1X_1 - X_1D_2)(i, j) = (2^{-i} - z_j)4^{-(i+j)}(2^{-i} - z_j)^{-1} = 4^{-(i+j)},$$

and so the range of $D_1X_1 - X_1D_2$ is the 1-dimensional subspace spanned by the vector $(1, 1/4, 1/16, \dots)$. Therefore, since

$$(D_1^*X_1 - X_1D_2^*)(i, j) = 4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j)),$$

it is clear that $D_1^*X_1 - X_1D_2^*$ has an infinite-dimensional range provided that we choose the sequence (z_n) with one additional property. That is, for each positive integer N , the N vectors given by $4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j))_{i=1}^\infty$ for $1 \leq j \leq N$ form a linearly independent set. Clearly, for this to hold, it is sufficient that these N vectors be linearly independent in the first N coordinates. Equivalently, it is sufficient to show that the determinant of the corresponding $N \times N$ matrix be nonzero; that is, it suffices to prove that there exists a sequence (z_n) in the closed left half-plane for which $|z_n| < 2^{-n}$ for every n and for which

$$\det(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j)))_{i,j=1}^N \neq 0$$

for every N . We prove this by induction.

Let $z_1 = 0$. Then the case $N = 1$ is trivial. (Observe that $N = 2$ means $\dim H = 4$; it is curious to note and easy to prove that if $\dim H = 3$ then $\text{rank}(DX - XD) = \text{rank}(D^*X - XD^*)$.)

Assume $\langle z_n \rangle_{n=1}^N$ has been chosen to satisfy the induction hypothesis. Let z_{N+1} denote the free complex variable which ranges over the intersection of the open, left half-plane and the open disc $[|z| < 2^{-(N+1)}]$. Then

$$\begin{aligned} f(z_{N+1}, \bar{z}_{N+1}) &= \det(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j)))_{i,j=1}^{N+1} \\ &= \sum_{i=1}^{N+1} (-1)^{N+1-i} D_i 4^{-(i+N+1)} ((2^{-i} - \bar{z}_{N+1})/(2^{-i} - z_{N+1})), \end{aligned}$$

where D_i is the subdeterminant of the $(i, N+1)$ entry which is

$$4^{-(i+N+1)}((2^{-i} - \bar{z}_{N+1})/(2^{-i} - z_{N+1})).$$

Clearly, by inspection and the induction hypothesis,

$$D_{N+1} = \det(4^{-(i+j)}((2^{-i} - \bar{z}_j)/(2^{-i} - z_j)))_{i,j=1}^N \neq 0,$$

and we will use this fact presently. If we let $a_i = (-1)^{N+1-i} D_i 4^{-(i+N+1)}$ and $z = z_{N+1}$, then it suffices to show that there exists a z contained in the intersection of the open, left half-plane and the open disc $[|z| < 2^{-(N+1)}]$ such that

$$f(z, \bar{z}) = \sum_{i=1}^{N+1} a_i ((2^{-i} - \bar{z})/(2^{-i} - z)) \neq 0.$$

To see that such a complex number z exists, suppose to the contrary that $f(z, \bar{z}) = 0$ in this region. Taking the $\bar{z}$ derivative of both sides of this equation, we obtain $0 = f_{\bar{z}}(z, \bar{z}) = \sum_{i=1}^{N+1} -a_i/(2^{-i} - z)$, for every z in this region. However,

$a_{N+1} = D_{N+1}4^{-2(N+1)} \neq 0$ and so $f_z(z, \bar{z}) = \sum_{i=1}^{N+1} -a_i/(2^{-i} - z)$ becomes unbounded in the open disc $[|z| < 2^{-(N+1)}]$ near the point $z = 2^{-(N+1)}$. Therefore, $f_z(z, \bar{z})$ is not identically 0 in this open disc. But since $f_z(z, \bar{z}) = 0$ throughout the intersection of the open, left half-plane and the open disc, we must have that $f_z(z, \bar{z})$ is identically 0 in this open disc, which is a contradiction. Q.E.D.

We now present the result of the Schatten p -classes. Let T^2 denote the torus.

THEOREM 2. *The classes C_p , for $0 < p < 1$, do not possess the GFP. In fact, if $N = M_{z+w}$ acting on $L^2(T^2)$, then for each $0 < p < 1$, there exists $X \in C_p$ such that $NX - XN \in C_p$ and yet $N^*X - XN^* \notin C_p$.*

PROOF. It is well known that $\{z^i w^j\}_{i,j=-\infty}^{\infty}$ is an orthonormal basis for $L^2(T^2)$. In this basis, the matrix of M_{z+w} is $N = U \otimes I + I \otimes U$, where U denotes the bilateral shift. Therefore the matrix of M_{z+w} has nonzero blocks on the diagonal and lower diagonal, with all other blocks 0. The diagonal blocks are bilateral shifts and the lower diagonal blocks are the identity I .

We choose X to look exactly the same as M_{z+w} except that the diagonal blocks will be weighted shifts and the lower diagonal blocks will be diagonal matrices. Let y_{ij} denote the entries of the j th diagonal block weighted shift, where y_{ij} is the i th weight on its lower diagonal (i.e. in the $(i, i-1)$ position). In the j th lower diagonal block (in the j th block row) let x_{ij} denote the i th diagonal entry (i.e. in the (i, i) position).

By computing the matrix entries of the commutators $NX - XN$ and $N^*X - XN^*$ we obtain

$$\begin{aligned} \|NX - XN\|_p^p &\leq \sum_{i,j} |x_{ij} - x_{ij-1}|^p + |y_{ij} - y_{i-1j}|^p \\ &\quad + |(x_{ij} - x_{i-1j}) - (y_{ij} - y_{i-1j})|^p \end{aligned}$$

and

$$\|N^*X - XN^*\|_p^p \geq \sum_{i,j} |x_{ij} - x_{i-1j}|^p.$$

To accomplish that $\|NX - XN\|_p < \infty$ and $\|N^*X - XN^*\|_p = \infty$, it clearly suffices to obtain x_{ij}, y_{ij} uniformly bounded, doubly indexed sequences (which insures that $X \in L(H \otimes H)$) such that

- (1) $\sum_{i,j} |x_{ij} - x_{ij-1}|^p < \infty$,
- (2) $\sum_{i,j} |y_{ij} - y_{i-1j}|^p < \infty$,
- (3) $\sum_{i,j} |(x_{ij} - x_{i-1j}) - (y_{ij} - y_{i-1j})|^p < \infty$,
- (4) $\sum_{i,j} |x_{ij} - x_{i-1j}|^p = \infty$.

We shall construct matrix arrays for (x_{ij}) and (y_{ij}) by taking the direct sums of finite arrays. Let $a_n A_n$ and $b_n B_n$ denote the square finite matrix arrays to be constructed. The scalars a_n, b_n represent positive numbers we shall choose in order to guarantee that the entires of $\sum \oplus a_n A_n$ and $\sum \oplus b_n B_n$ remain uniformly bounded (to insure that X be a bounded operator) and also to guarantee that they

satisfy conditions (1)–(4). Choose A_n and B_n to be the matrix arrays

$$\begin{array}{cccccccc}
 & & & & A_n & & & \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\
 0 & 1 & 1 & 1 & \cdots & 1 & 0 & \\
 0 & 2 & 2 & 2 & \cdots & 2 & 0 & \\
 0 & 3 & 3 & 3 & \cdots & 3 & 0 & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 0 & n & n & n & \cdots & n & 0 & \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 &
 \end{array}
 \qquad
 \begin{array}{cccccccc}
 & & & & B_n & & & \\
 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\
 0 & 1 & 2 & \cdot & \cdots & n & 0 & \\
 0 & 1 & 2 & \cdots & n & 0 & & \\
 0 & 1 & 2 & \cdots & n & 0 & & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \\
 0 & 1 & 2 & \cdot & \cdots & n & 0 & \\
 0 & 0 & 0 & \cdot & \cdots & 0 & 0 &
 \end{array}$$

For convenience, let (1)–(4) also denote the corresponding sums in conditions (1)–(4). Let us investigate how (1)–(4) act on A_n , B_n where (x_{ij}) is A_n and (y_{ij}) is B_n (considering them as two-way infinite arrays by extending them to have all other entries 0). By computing quantities (1)–(4) we see that

$$\begin{aligned}
 (1) &= (2) = 2 \sum_{k=1}^n k^p < 2n \cdot n^p = 2n^{p+1}, \\
 (3) &= 2 \cdot n \cdot n^p = 2n^{p+1}, \\
 (4) &= \sum_{n^2 \text{ times}} 1 + n \cdot n^p > n^2.
 \end{aligned}$$

In other words, (1), (2) and (3) applied to A_n , B_n have orders of magnitude of n^{p+1} , whereas (4) grows with an order of magnitude of n^2 . The clincher is that $p + 1 < 2$ (since $p < 1$). Set $a_n = 1/n^{3/2} = b_n$. Clearly $a_n, b_n < 1/n$ and so the entries of $a_n A_n$ and $b_n B_n$ are all less than or equal to 1. Furthermore, applying (1)–(4) to $a_n A_n$ and $b_n B_n$ we obtain

$$\begin{aligned}
 (1) &< (1/n^{3/2})^2 \cdot 2n^{p+1} = 2/n^{2-p} \quad \text{and similarly,} \\
 (2) &< 2/n^{2-p}, \\
 (3) &< 2/n^{2-p}, \\
 (4) &\geq n^2/n^3 \geq 1/n.
 \end{aligned}$$

Now take (x_{ij}) to be the two-way infinite matrix array $\Sigma \oplus a_n A_n$ and (y_{ij}) to be $\Sigma \oplus b_n B_n$. However here we must just consider $a_n A_n$ and $b_n B_n$ as finite blocks surrounded by enough zeros so that when “pasted” together in the matrix they do not overlap and so (1)–(4) remain as before. The quantities (1)–(4) applied to $\Sigma \oplus a_n A_n$ and $\Sigma \oplus b_n B_n$ are simply the sum of the corresponding quantities (1)–(4) respectively, applied to $a_n A_n$, $b_n B_n$. This is because the sums (1)–(4) act independently on each finite block array in $\Sigma \oplus a_n A_n$ and $\Sigma \oplus b_n B_n$. Hence since

$2 - p > 1$, we have

$$(1) \leq \sum \frac{2}{n^{2-p}} < \infty, \quad (2) \leq \sum \frac{2}{n^{2-p}} < \infty,$$

$$(3) \leq \sum \frac{2}{n^{2-p}} < \infty, \quad (4) \geq \sum \frac{1}{n} = \infty. \quad \text{Q.E.D.}$$

REMARK. These investigations have led to another kind of possible generalization of Fuglede's Theorem. Namely, let $\langle M_k \rangle$ and $\langle N_k \rangle$ be commuting sequences of normal operators (i.e., $M_k M_j = M_j M_k$ and $N_k N_j = N_j N_k$). Let $X \in L(H)$. The question is: Does $\sum_{k=1}^N M_k X N_k = 0$ imply $\sum_{k=1}^N M_k^* X N_k^* = 0$? Also if $\sum \|M_k\| \|N_k\| < \infty$, does $\sum_1^\infty M_k X N_k = 0$ imply $\sum_1^\infty M_k^* X N_k^* = 0$? Also are the corresponding statements true modulo the Hilbert-Schmidt class?

Thus far we know easily that all answers are yes when all normal operators are diagonalizable, and C. Apostol settled the questions in the affirmative in the case $N = 2$ using [6].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221