WHEN ARE PROPER SUBGROUPS OF LCA GROUPS CONTAINED IN MAXIMAL ONES?

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ABSTRACT. In this note we determine (1) the class of LCA groups for which every proper closed subgroup is contained in a maximal subgroup, (2) the class of LCA groups for which every proper dense subgroup is contained in a maximal subgroup, and (3) the class for which every proper subgroup is contained in a maximal one. We also determine when both an LCA group and its dual have these properties.

1. Introduction. Let G be an LCA group. We shall say that G possesses property C if every proper closed subgroup of G is contained in a maximal subgroup. G possesses property D if every proper dense subgroup is contained in a maximal subgroup, and G has property E if every proper subgroup is contained in a maximal one. $B(G)$ denotes the subgroup of compact elements; $D(G)$, a minimal divisible extension of G , topologized in the usual manner so that G is an open subgroup. G_0 is the component of the neutral element, and $\hat{G}$, the character group of G . The symbol " $\cong$ " denotes topological isomorphism, and " $\approx$ " denotes isomorphism in the algebraic sense.

For an abelian group A (topological or not), $T(A)$ and $U(A)$ denote respectively the maximal torsion subgroup and the first Ulm subgroup of A (see [3]). $r(A)$ is the rank of A . A subgroup of A is a maximal subgroup iff it is of prime index. We define an abelian group A to be totally reduced if every nontrivial homomorphic image of A is reduced. As we show below, totally reduced abelian groups provide just the right kind of group-theoretic machinery to tackle the problems of this paper.

We shall use the quasicyclic groups $Z(p^\infty)$ and the group of additive rationals $\mathbb{Q}$. The well-known LCA groups which we shall mention are the circle group T , the additive real numbers R , the additive group of p -adic integers J_p (all with their usual topologies), and the discrete group of integers Z . For other notation and terminology, we follow [3] and [4].

Finally, we would like to point out that the material in this paper is related to some structural and measure-theoretic properties of LCA groups discussed in [2], [5], [6] and [7], and that partial results concerning the questions raised herein have appeared in [6, Theorem 1.10 and Corollary 1.11].

Received by the editors November 19, 1980.

1980 *Mathematics Subject Classification*. Primary 22B05; Secondary 20K27, 20K99.

Key words and phrases. Maximal subgroups, reduced, totally reduced, divisible, abelian groups, locally compact abelian, p -adic integers, quasicyclic.

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 0002-9939/81/0000-0430/\$02.75

2. Results. Our first object is to determine the structure of totally reduced abelian groups. We start with a result that brings out the relevance of this class of groups to the theme of this paper.

PROPOSITION 1. *Every proper subgroup of an abelian group is contained in a maximal subgroup iff it is totally reduced.*

PROOF. Let A be totally reduced and let H be a proper subgroup. Since A/H is reduced, it contains a maximal subgroup [3, Exercise 1, p. 99]. Hence, H is contained in a maximal subgroup of A . Conversely, let L be a proper subgroup of A such that A/L is not reduced. Then L is contained in a proper subgroup M such that A/M is divisible, and this subgroup M is not contained in any maximal subgroup. Hence, A must be totally reduced, and the proof is complete.

Having done this, we can now move towards our first goal with a clearer conscience.

PROPOSITION 2. *Subgroups and quotient groups of totally reduced abelian groups are totally reduced.*

PROOF. Let A be totally reduced and let H be a proper subgroup of A . If H is not totally reduced, H contains a proper subgroup K such that H/K is divisible. Since $A/H \approx (A/K)/(H/K)$, it follows that A/K is not reduced, contradicting the hypothesis. Hence, H is totally reduced. The assertion about quotient groups follows from the definition of totally reduced groups.

COROLLARY 1. *An abelian group A is totally reduced iff both $T(A)$ and $A/T(A)$ are totally reduced.*

PROOF. The “necessity” part follows from Proposition 2. For the “sufficiency”, suppose $T(A)$ and $A/T(A)$ are totally reduced, and H is an arbitrary proper subgroup of A . If $H + T(A) = A$, then $A/H \approx T(A)/H \cap T(A)$, and so is reduced. If $H + T(A) \neq A$, then $A/(H + T(A))$ is a homomorphic image of $A/T(A)$, and so must be reduced. This implies that $H + T(A)$ is contained in a maximal subgroup of A . Clearly the same is true for H , and the proof is complete.

In view of Corollary 1, it suffices to determine the structure of those totally reduced groups which are either torsion or torsion-free.

PROPOSITION 3. *Let A be an abelian group.*

(a) *If A is a torsion group, then A is totally reduced iff each p -component of A is bounded.*

(b) *If A is torsion-free, then A is totally reduced iff $r(A)$ is finite, and for every free subgroup M of A such that $r(A) = r(M)$, each p -component of A/M is finite.*

PROOF. (a) Suppose a p -component A_p of A is not bounded. Then A_p contains a proper basic subgroup, say H , and A_p/H is divisible [3, Corollary 35.4]. But this implies that A is not totally reduced. Conversely, if A_p is bounded for every prime p , then no homomorphic image of A is $\approx Z(p^\infty)$ for any prime p . Hence, in this case, A is totally reduced.

(b) First let A be totally reduced. Suppose, if possible, that $r(A)$ is infinite and let N be a free subgroup generated by a countably infinite independent subset of A . By an obvious construction, we can get a subgroup S of N such that $N/S \approx \bigoplus_{k=1}^{\infty} Z(p^k)$. As pointed out in the proof of (a), this group has a proper basic subgroup and so Proposition 2 implies that A is not totally reduced. Hence, $r(A)$ must be finite. Now, let M be a free subgroup with $r(M) = r(A)$. Then for every prime p , the p -component of A/M is a reduced subgroup of a direct sum of finitely many copies of $Z(p^{\infty})$. Hence, it is finite. Conversely, suppose A has a nonzero divisible homomorphic image. Then A has a nonzero torsion divisible homomorphic image, A/H say, where H contains a free subgroup, say M , with $r(A) = r(M)$. But this contradicts the hypothesis about the p -components of A/M . Hence, A is totally reduced and the proof is complete.

Using duality, we deduce a result of some interest about compact abelian groups.

PROPOSITION 4. *A compact abelian group G contains no nontrivial closed torsion-free subgroup iff $\hat{G}$ is totally reduced.*

PROOF. The discrete group $\hat{G}$ is totally reduced iff it has no homomorphic image $\approx Z(p^{\infty})$ for any prime p . By [4, 25.8], a compact abelian group contains no nontrivial closed torsion-free subgroup iff it contains no copy of J_p for any prime p . The result follows by duality.

We are now in business. It is clear that an LCA group has property E iff it has properties C and D both. We could deal with properties C and D first, and use them to study property E. However, a direct treatment of property E via totally reduced groups is simpler and more natural. First, we make a remark whose truth is obvious, and which we shall use in the sequel.

REMARK 1. Let G be an LCA group. Then $T(G)$ is open iff G_0 is trivial and $T(G) = B(G)$.

We are now ready for our first theorem.

THEOREM 1. *An LCA group G possesses property E iff $T(G)$ is open and $\approx M \times N$, where M is a bounded LCA group, N is a discrete, torsion, totally reduced group, and $G/T(G)$ is a (discrete torsion-free) totally reduced group.*

PROOF. By Proposition 1, it follows that an LCA group has property E iff it is totally reduced. Now, G is clearly totally reduced in case the structure of G is as given in the hypothesis. Conversely, let G be totally reduced. Then G is totally disconnected. If $T(G)$ is a proper subgroup of $B(G)$, then $B(G)$ contains a compact monothetic (necessarily uncountable) subgroup $H \cong$ a topological direct product $\prod_p H_p$, where H_p is either J_p or a cyclic p -group or trivial [4, 25.16]. Since H is uncountable and $T(H)$ is countable, H is not totally reduced. Hence, $T(G) = B(G)$, so that $T(G)$ is open by Remark 1. Also $G/T(G)$ is totally reduced. Let $\bigoplus_p G_p$ be the decomposition of $T(G)$ into p -components as an abstract group. Let K be a compact, open, n -bounded subgroup. Let M be the (open) subgroup generated by those p -components of $T(G)$ for which p divides n , and let N be the discrete

complement of M . Since M is open, we conclude that $T(G) \cong M \times N$, and the proof is complete.

As remarked in the proof, Theorem 1 describes the structure of totally reduced LCA groups.

The following two corollaries follow directly from Theorem 1.

COROLLARY 2. *A torsion-free LCA has property E iff it is discrete and totally reduced.*

COROLLARY 3. *A compact abelian group has property E iff it is torsion (hence, bounded).*

We turn next to property D.

THEOREM 2. *An LCA group G has property D iff $T(G)$ is open.*

PROOF. Suppose $T(G)$ is open and let H be a proper dense subgroup of G . $T(G)$ contains a compact, open, bounded subgroup K , say $mK = \{0\}$. Then H contains mG [5, Lemma 5.5]. Since G/mG is totally reduced, it is obvious that H is contained in a maximal subgroup of G .

Conversely, suppose G has property D. Then T cannot be a homomorphic image of G , for this would contradict property D. Hence, G is totally disconnected. In view of Remark 1, it is enough to show that $T(G) = B(G)$. Suppose not, then $B(G)$ contains an infinite cyclic subgroup. Hence $D(G)$ contains a copy of $\mathbb{Q}$ with the nondiscrete topology. By [1, Theorem 2], $\mathbb{Q}$ is divisible. Hence, we have $D(G) = \overline{\mathbb{Q}} + L$, for some subgroup L . Now $(\mathbb{Q} + L) \cap G$ is clearly a proper dense subgroup of G , and

$$G/((\mathbb{Q} + L) \cap G) \approx (G + \mathbb{Q} + L)/(\mathbb{Q} + L) = D(G)/(\mathbb{Q} + L),$$

which is divisible. This contradicts property D. Hence, $T(G) = B(G)$ and the proof is complete.

COROLLARY 4. *If an LCA group G has property D, and H is a closed subgroup, then both H and G/H have property D.*

PROOF. For H , we note that $T(H) = H \cap T(G)$. For G/H , we observe that the image of the open subgroup $H + T(G)$ under the natural homomorphism onto G/H is contained in $T(G/H)$. Theorem 2 completes the proof.

COROLLARY 5. *A compact abelian group has property D iff it is bounded.*

PROOF. If G is compact, $T(G)$ is open iff $G = T(G)$.

COROLLARY 6. *Every torsion LCA group has property D.*

PROOF. Obvious.

REMARK 2. Theorem 2 needs some explanation. Suppose G is a torsion, divisible LCA group. Then G is as far from being totally reduced as it can be; it just contains no maximal subgroups. How then do the proper dense subgroups of G manage to be contained in maximal subgroups? Well, just by not existing. In fact, a

nondiscrete LCA group G contains no proper dense subgroups iff $T(G)$ is open and pG is open for every prime p [5, Corollary 5.1].

Before passing on to our next theorem, we insert a proposition, which will simplify our proofs.

PROPOSITION 5. *An LCA group contains no nontrivial closed torsion-free subgroup iff it contains no copy of $\mathbb{Z}$ or J_p .*

PROOF. A torsion-free LCA group is of the form $R^n \times M \times N$, where M is a topological direct product of copies of $(Q_d)^\wedge$ (where Q_d is Q taken discrete), and N is a totally disconnected torsion-free group [4, 25.30(c) and 25.8]. The proof is now obvious.

THEOREM 3. *The following are equivalent for an LCA group G :*

- (a) G has property C.
- (b) G is totally disconnected, and G/H is totally reduced for every open subgroup H of G .
- (c) $\hat{G}$ contains no nonzero closed torsion-free subgroup.
- (d) G is totally disconnected and contains a compact, open subgroup K such that G/K is totally reduced.

PROOF. (a) implies (b): First, G_0 must be trivial, for otherwise T would be a homomorphic image of G , contradicting (a). The assertion about open subgroups is obvious.

(b) implies (c): Since every element of $\hat{G}$ is compact, it is enough to show that $\hat{G}$ contains no copy of J_p [Proposition 5]. But this is clear because G has no discrete homomorphic image $\approx \mathbb{Z}(p^\infty)$.

(c) implies (d): Since $\hat{G}$ contains no copy of $\mathbb{Z}$, G is totally disconnected. Let L be a compact, open subgroup of $\hat{G}$ and let K be the annihilator of L in G . Then K is a compact, open subgroup such that G/K is totally reduced [Proposition 4].

(d) implies (a): Let H be a proper closed subgroup of G . If $H + K$ is a proper subgroup of G , then by the condition on K , $H + K$ is contained in a maximal subgroup, and so is H . If $H + K = G$, then $G/H \cong K/(H \cap K)$ by [4, 5.33]. Now $K/(K \cap H)$ is compact and totally disconnected, so cannot be divisible. This implies that H is contained in a maximal subgroup, and the proof is complete.

COROLLARY 7. *A torsion LCA group has property C iff it has property E.*

PROOF. Since every torsion LCA group has property D [Corollary 6], the assertion is obvious.

COROLLARY 8. *A compact abelian group has property C iff it is totally disconnected.*

PROOF. By (c) of Theorem 3.

J_p has property C, but is not totally reduced. However, we show below that if G has property C, then $\bigcap_{n=1}^{\infty} nG$ is trivial. In particular, G is reduced. First, we prove a lemma.

LEMMA 1. *If A is a totally reduced abelian group, then $U(A)$ is trivial.*

PROOF. Suppose not. If there is a $U(A)$ -high subgroup H , then A/H would be divisible [3, Exercise 12, p. 116]. Hence, $U(A)$ is an essential subgroup. By Proposition 3 and [3, Remark (c), p. 98], we conclude that A must be torsion-free. But then $U(A)$ is divisible [3, Exercise 2, p. 101]. Hence, $U(A)$ must be trivial.

COROLLARY 9. *If an LCA group G has property C, then $\bigcap_{n=1}^{\infty} \overline{nG}$ is trivial.*

PROOF. By [4, 24.24], it suffices to show that $\hat{G}$ contains a dense torsion subgroup. Since $\hat{G} = B(\hat{G})$, it is enough to show that the (compact) closure of every infinite cyclic subgroup of $\hat{G}$ contains a dense torsion subgroup. Let H be such a subgroup. Since $\hat{H}$ is discrete and totally reduced, we conclude by Lemma 1 that H does contain a dense torsion subgroup. This completes the proof.

COROLLARY 10. *If an LCA group G has property C, and H is a closed subgroup, then both H and G/H have property C.*

PROOF. By Theorem 3(c), it is clear that G/H has property C. Next, suppose K is a proper closed subgroup of H such that H/K is divisible. Then G/K contains a nonzero divisible subgroup, contradicting Corollary 9. Hence, H must have property C.

In the remainder of this note, we investigate when both G and $\hat{G}$ possess one or more of these properties.

THEOREM 4. *Let G be an LCA group. Then both G and $\hat{G}$ have property C iff G_0 is trivial, $G = B(G)$, and G has a compact open subgroup H such that both G/H and $\hat{H}$ are totally reduced.*

PROOF. Suppose both G and $\hat{G}$ satisfy the property, then G and $\hat{G}$ are totally disconnected [Theorem 3], so $G = B(G)$. Also G has a compact open subgroup H such that G/H is totally reduced [Theorem 3(d)]. Now, $A(\hat{G}, H)$ is an open subgroup of $\hat{G}$, so $\hat{G}/A(\hat{G}, H)$ is totally reduced [Theorem 3(b)]. But this implies $\hat{H}$ is totally reduced. Conversely, the hypothesis implies that both G and $\hat{G}$ are totally disconnected, and that condition (d) of Theorem 3 is satisfied by both G and $\hat{G}$. This completes the proof.

Before passing on to the last theorem of this note, we make an obvious but useful remark.

REMARK 3. Let G be an LCA group. Then both G and $\hat{G}$ are torsion groups iff G is bounded. Moreover, in this case G is n -bounded iff $\hat{G}$ is n -bounded.

THEOREM 5. *The following are equivalent for an LCA group G .*

- (a) *Both G and $\hat{G}$ have property D.*
- (b) *Both G and $\hat{G}$ have property E.*
- (c) *G is bounded.*

PROOF. Using Theorems 1 and 2 and Remark 3, the proof follows immediately.

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