STRATIFIABLE SPACES AS SUBSPACES AND CONTINUOUS IMAGES OF M_1 -SPACES

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ABSTRACT. It is shown that every stratifiable space is the image of an M_1 -space under a perfect retraction.

In 1961, J. Ceder introduced three classes of generalized metric spaces that he called M_i -spaces, $i = 1, 2, 3$. Ceder showed that every M_1 -space is an M_2 -space and every M_2 -space is an M_3 -space, and he asked whether the converse results also hold [3]. C. Borges renamed M_3 -spaces "stratifiable spaces" and proved several important results concerning these spaces in [1]. One half of Ceder's question was answered by G. Gruenhage and the second author of this paper, who showed, independently, that every stratifiable space is an M_2 -space ([4] and [6]); the other half of the question has so far remained unanswered (for some partial answers, see [8] and [5]).

While the precise relationship between stratifiable spaces and M_1 -spaces is yet to be determined, the following result sheds some light on that relationship.

THEOREM. *Every stratifiable space is the image of an M_1 -space under a perfect retraction.*

PROOF. Let X be a stratifiable space. We assume that X is a T_1 -space; it is easy to modify the following proof so that it works also for non- T_1 -spaces (note that stratifiable spaces are regular and hence "essentially T_1 ").

Denote by S the subspace $\{0\} \cup \{1/n \mid n \in N\}$ of the real line, and denote by Z the space whose ground-set is the product-set $X \times S$ and whose topology is obtained by enlarging the product topology so as to make every point of the set $X \times \{1/n \mid n \in N\}$ isolated.

It is not difficult to see that Z is an M_1 -space and that the projection $X \times S \rightarrow X$ is a retraction from Z onto X ; however, to obtain a perfect retraction, we must construct a suitable subspace of Z . Since X is an M_2 -space, there exists a family $\mathcal{F} = \bigcup_{n \in N} \mathcal{F}_n$ of closed subsets of X such that every point of X has a neighborhood base consisting of members of $\mathcal{F}$ and, for every $n \in N$, the family $\mathcal{F}_n$ is closure-preserving. We may assume $X \in \mathcal{F}_1$. It follows from Theorems 3.14 and 4.8 of [6] that, for every $n \in N$, there exists a family $\mathcal{K}_n = \bigcup_{k \in N} \mathcal{K}_{n,k}$ of subsets of X such that, for each $k \in N$, the family $\mathcal{K}_{n,k}$ is discrete, and, for each $x \in X$, there

Received by the editors April 25, 1980.

1980 *Mathematics Subject Classification.* Primary 54E20, 54C15, 54C25; Secondary 54C10, 54B55.

¹While working on this paper, the second author was visiting the University of Pittsburgh as an Andrew Mellon Postdoctoral Fellow.

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 0002-9939/81/0000-0433/\$01.75

exists $K \in \mathcal{K}_n$ such that $x \in K \subset \bigcap \{F \in \bigcup_{m=1}^n \mathcal{F}_m \mid x \in F\}$. For each $K \in \bigcup_{n \in N} \mathcal{K}_n$, let x_K be an element of K . For every $n \in N$, let

$$\mathcal{K}_n = \bigcup \{\mathcal{K}_{m,k} \mid m \leq n \text{ and } k \leq n\},$$

and let $A_n = \{x_K \mid K \in \mathcal{K}_n\}$; note that $\mathcal{K}_n$ is locally finite and, consequently, A_n is discrete and closed. Let Y be the subspace $(X \times \{0\}) \cup \bigcup_{n \in N} (A_n \times \{1/n\})$ of Z .

We show that Y is an M_1 -space. For all $F \in \mathcal{F}$ and $k \in N$, let

$$F_k = [(\text{Int}_X F) \times \{0\}] \cup [Y \cap (F \times \{1/n \mid n \geq k\})].$$

We show that, for all $x \in X$, $F \in \mathcal{F}$ and $k \in N$, if $x \in F$, then $(x, 0) \in \text{Cl}_Y F_k$. Let $F \in \mathcal{F}$ and $k \in N$, and let $x \in F$. To show that $(x, 0) \in \text{Cl}_Y F_k$, assume on the contrary that $(x, 0) \notin \text{Cl}_Y F_k$. Then there exists a neighborhood G of x in X and $m \in N$ such that $(G \times \{1/n \mid n \geq m\}) \cap F_k = \emptyset$. Let $E \in \mathcal{F}$ be such that $x \in E \subset G$, and let $h \in N$ be such that $\{E, F\} \subset \bigcup_{n=1}^h \mathcal{F}_n$. There exists $K \in \mathcal{K}_h$ such that $x \in K \subset E \cap F$. Let $i \in N$ be such that $K \in \mathcal{K}_{h,i}$, and let $j = h + i + k + m$. Then $(x_K, 1/j) \in F_k$ and $(x_K, 1/j) \in G \times \{1/n \mid n \geq m\}$; this, however, is a contradiction since $(G \times \{1/n \mid n \geq m\}) \cap F_k = \emptyset$. It follows that $(x, 0) \in \text{Cl}_Y F_k$.

For every $n \in N$, let $\mathcal{B}_n = \{F_k \mid F \in \mathcal{F}_n \text{ and } k \in N\}$. We show that, for each $n \in N$, the family $\mathcal{B}_n$ is closure-preserving in Y . Let $n \in N$ and $\mathcal{B} \subset \mathcal{B}_n$, and let $y \in \text{Cl}_Y \bigcup \mathcal{B}$. If $y \in X \times \{1/n \mid m \in N\}$, then y is isolated and, consequently, $y \in \bigcup \mathcal{B}$. Assume that $y = (x, 0)$ for some $x \in X$. Let

$$G = X \sim \bigcup \{F \in \mathcal{F}_n \mid x \notin F\}.$$

Since $\mathcal{F}_n$ is closure-preserving in X , the set G is a neighborhood of x in X , and hence the set $G' = Y \cap (G \times S)$ is a neighborhood of $(x, 0)$ in Y . Since $(x, 0) \in \text{Cl}_Y \bigcup \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B \cap G' \neq \emptyset$. Let $F \in \mathcal{F}_n$ and $k \in N$ be such that $B = F_k$. We have $F_k \cap G' \neq \emptyset$ and consequently, $F \cap G \neq \emptyset$, that is $x \in F$. From the preceding part of the proof it follows that $(x, 0) \in \text{Cl}_Y F_k$; in other words, that $y \in \text{Cl}_Y B$. We have shown that the families $\mathcal{B}_n$ are closure-preserving in Y . Clearly, the family $\mathcal{B}_0 = \{\{y\} \mid y \in Y(X \times \{1/n \mid n \in N\})\}$ is σ -closure-preserving in Y . It is easily seen that the family $\bigcup_{n=0}^{\infty} \mathcal{B}_n$ is a base for the topology of Y . Since X is a regular space, so is Y . Consequently, Y is an M_1 -space.

To complete the proof, denote by f the restriction of the projection map $X \times S \rightarrow X$ to Y . It is easily seen that f is a continuous mapping from Y onto X and that for each $x \in X$, the set $f^{-1}\{x\}$ is compact in Y . We show that the mapping f is closed. Let C be a closed subset of Y , and let $x \in X \sim f(C)$. Then $C \cap f^{-1}\{x\} = \emptyset$ and hence $(x, 0) \notin C$. Consequently, there exists a neighborhood U of x in X and $m \in N$ such that $[U \times (\{0\} \cup \{1/n \mid n \geq m\})] \cap C = \emptyset$. For each $n < m$, since $(x, 1/n) \notin C$ and since A_n is a closed and discrete subspace of X , the set $V_n = X \sim \{a \in A_n \mid (a, 1/n) \in C\}$ is a neighborhood of x in X . It is easily seen that the neighborhood $U \cap \bigcup_{n < m} V_n$ of x is disjoint from the set $f(C)$. We have shown that the mapping f is closed; hence f is a perfect mapping. Setting $g(x) = (x, 0)$ for each $x \in X$, we obtain a homeomorphism between X and the subspace $X' = X \times \{0\}$ of Y . That f is a retraction follows by observing that the restriction of the mapping $g \circ f$ to X' is the identity map on X' .

Since stratifiability is a hereditary property [3] and since the continuous image of a stratifiable space under a closed mapping is stratifiable [1], it follows from the above theorem that the problem, whether every stratifiable space is an M_1 -space, is equivalent with some problems concerning the preservation of the M_1 -property in topological operations.

COROLLARY. *The following statements are mutually equivalent.*

- (i) *Every stratifiable space is an M_1 -space.*
- (ii) *For every M_1 -space, every closed subspace of the space is an M_1 -space.*
- (iii) *For every M_1 -space, all the images of the space under perfect mappings are M_1 -spaces.*

The implication (ii) $\Rightarrow$ (iii) in the Corollary also follows from the result of C. Borges and D. Lutzer that if every closed subspace of a space is an M_1 -space, then every image of the space under a perfect mapping is an M_1 -space [2].

Let us call a topological space an M_0 -space if the topology of the space has a σ -closure-preserving base consisting of sets that are both open and closed. Every M_0 -space is a strongly 0-dimensional M_1 -space. It is easily seen that a subspace of an M_0 -space is again an M_0 -space; from the result of Borges and Lutzer mentioned above it follows that the image of an M_0 -space under a perfect mapping is an M_1 -space. It is not known whether every M_1 -space can be represented as the image of an M_0 -space under a perfect mapping; for metrizable spaces, such a representation is obtained [7]. Note that it follows from the Theorem that if every M_1 -space is the image of an M_0 -space under a perfect mapping, then every stratifiable space is an M_1 -space.

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