FLAT BUNDLES WITH SOLVABLE HOLONOMY. II. OBSTRUCTION THEORY

WILLIAM M. GOLDMAN

ABSTRACT. Necessary and sufficient conditions for a connected solvable Lie group G are given so that every flat principal G -bundle over a CW-complex is trivial after passing to a finite covering space.

In [2] it was proved that if ξ is a flat principal G -bundle over a CW-complex M where G is a solvable linear Lie group then there is a finite-sheeted (Galois) covering $p: \hat{M} \rightarrow M$ such that $p^*\xi$ is trivial. (If such a covering p exists we say that ξ is *virtually trivial*.) In this note we extend this result to general solvable Lie groups G , giving necessary and sufficient conditions for the above conclusion. In particular we show that there exist flat principal G -bundles which are not virtually trivial.

THEOREM. *Let G be a connected solvable Lie group. Then the following conditions are equivalent:*

- (a) *The commutator subgroup G' of G is simply connected (in its Lie group topology);*
- (b) *every flat principal G -bundle over a CW-complex is virtually trivial.*

PROOF. (a) $\Rightarrow$ (b). Suppose G' is simply connected and ξ is a flat principal G -bundle over M . We shall show that all the obstructions to trivializing ξ vanish after pulling back over a finite covering. (See Steenrod [7] for a description of the method of obstruction theory.) The first obstruction lies in $H^1(M; \pi_0(G))$ and therefore vanishes since G is connected. The next obstruction (which we denote $o_2(\xi)$) lies in $H^2(M; \pi_1(G))$. Let $\pi = \pi_1(M)$ and let $\phi: \pi \rightarrow G$ be the holonomy homomorphism of ξ . Then the obstruction $o_2(\xi)$ is simply the obstruction to lifting ϕ to a homomorphism $\pi \rightarrow \tilde{G}$, where $\tilde{G}$ denotes the universal cover of G . Although in general $o_2(\xi)$ does not vanish, we will show that it is a torsion class.

We begin by showing that for every homology class $z \in H_2(M; \mathbb{Z})$, the product $o_2(\xi) \cdot z \in \pi_1(G)$ is zero. Since ξ is a flat bundle, its classifying map factors through the Eilenberg-Mac Lane space of type $K(\pi, 1)$. Indeed its classifying map may be taken to be the composition $M \xrightarrow{f} K(\pi, 1) \xrightarrow{B\phi} K(G, 1)$ where f is a map inducing an isomorphism on fundamental groups and $B\phi$ is the map induced by ϕ . Thus ξ is induced by a flat principal G -bundle over $K(\pi, 1)$ and we may assume with no loss of generality that M is a space of type $K(\pi, 1)$.

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In Hopf [4] the following description of $H_2(M; Z)$ is given when M is a $K(\pi, 1)$. Write π as a quotient F/R where F is a free group on some generators $f_1, \dots, f_k$ and R is a normal subgroup of F . Then $H_2(M; Z)$ is $(R \cap [F, F])/[F, R]$ where $[A, B]$ denotes the group generated by all commutators $[a, b] = a^{-1}b^{-1}ab$ where $a \in A, b \in B$. The above formula is proved by showing that each homology class $z \in H_2(M; Z)$ is represented by a map of a closed orientable surface F into M ; such maps are represented (up to homotopy) by homomorphisms $\pi_1(F) \rightarrow \pi$. Using the well-known presentation of the fundamental group of a surface $\pi_1(F) = \langle a_1, b_1, \dots, a_n, b_n | [a_1, b_1] \cdots [a_n, b_n] = 1 \rangle$ one sees that homology classes z correspond to relations in π of the form $[A_1, B_1] \cdots [A_n, B_n] = 1$ where the A_i, B_i are themselves words in the generators of π .

Let $R = R(g_1, \dots, g_n)$ be a relation in the generators $g_1, \dots, g_n$ of π such that under the above correspondence R corresponds to the homology class z . Then $o_2(\xi) \cdot z$ may be computed as follows. Choose lifts $\phi(g_i)^\sim$ of the $\phi(g_i)$ to $\tilde{G}$; then consider the element $R(\phi(g_1)^\sim, \dots, \phi(g_n)^\sim)$ of $\tilde{G}$. Since it is a product of commutators and choices of lifts $\phi(g_i)^\sim$ differ by elements of the center of $\tilde{G}$, this element is independent of the choice of lifts. Moreover since $R(g_1, \dots, g_n) = 1$, the constructed element of $\tilde{G}$ lies in the kernel of $\tilde{G} \rightarrow G$ which we identify with $\pi_1(G)$. It is not difficult to verify (see Milnor [5] and Hirzebruch [8, §I]) that $R(\phi(g_1)^\sim, \dots, \phi(g_n)^\sim) \in \pi_1(G)$ equals $o_2(\xi) \cdot z$.

Now R is a product of commutators, so $o_2(\xi) \cdot z$ lies in the commutator subgroup $(\tilde{G})'$ of $\tilde{G}$. On the other hand it lies in $\pi_1(G) \subset \tilde{G}$. But $\pi_1(G) \cap (\tilde{G})'$ is the fundamental group of G' , so that $o_2(\xi) \cdot z = 0$ for all $z \in H_2(M; Z)$.

Recall the universal coefficient exact sequence

$$0 \rightarrow \text{Ext}(H_1(M; Z), \pi_1(G)) \rightarrow H^2(M; \pi_1(G)) \rightarrow \text{Hom}(H_2(M; Z), \pi_1(G)) \rightarrow 0.$$

We have proved that the image of $o_2(\xi)$ in $\text{Hom}(H_2(M; Z), \pi_1(G))$ is zero; by exactness $o_2(\xi)$ lies in the image of $\text{Ext}(H_1(M; Z), \pi_1(G))$ which is a finite abelian group. It follows that $o_2(\xi)$ is a torsion class.

To see that ξ is virtually trivial, let T denote the torsion part of $H_1(M; Z)$ and let $p: \hat{M} \rightarrow M$ be the covering space of M corresponding to the kernel of the natural surjective homomorphism $\pi_1(M) \rightarrow T$. Since T is a finite group, p is a finite-sheeted covering space. Working through the definitions one sees that the obstruction class $o_2(p^*\xi)$ is the image of $k \cdot o_2(\xi)$ under $p^*: H^2(M; \pi_1(G)) \rightarrow H^2(\hat{M}; \pi_1(G))$ where k equals the order of T . Since $k \cdot o_2(\xi) = 0$, it follows that $o_2(p^*\xi) = 0$.

The next obstructions for trivializing $p^*\xi$ all lie in cohomology groups $H^{i+1}(\hat{M}; \pi_i(G))$ for $i \geq 2$. As G is a solvable Lie group, $\pi_i(G) = 0$ so all the successive obstructions for $p^*\xi$ vanish. Therefore we have proved that $p^*\xi$ is trivial.

(b) $\Rightarrow$ (a). Suppose that G' is not simply connected; we will find a principal G -bundle over a surface which is not virtually trivial. Since G' is not simply connected, we may find a nontrivial element $c \in (\tilde{G})' \cap \pi_1(G) \subset \tilde{G}$. Thus there exist elements $\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b}_1, \dots, \tilde{b}_n \in \tilde{G}$ such that $[\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_n, \tilde{b}_n] = c$. Denote their images in G by $a_1, \dots, a_n, b_1, \dots, b_n$; then we may construct a

homomorphism $\phi: \pi_1(F) \rightarrow G$, where F is a closed orientable surface of genus n , by sending the usual system of generators for $\pi_1(F)$ to the a_i, b_i . We claim the associated flat principal G -bundle ξ over F is not virtually trivial. The obstruction to trivializing ξ is the element corresponding to c in $H^2(F; \pi_1(G)) \simeq \pi_1(G)$ and is nonzero since the map on fundamental groups induced by the inclusion $G' \rightarrow G$ is injective. Moreover if $p: \hat{F} \rightarrow F$ is a k -fold covering then the obstructions satisfy $o_2(p^*\xi) = p^*o_2(\xi)$. There is a commutative diagram

$$\begin{array}{ccc} H^2(F; \pi_1(G)) & \simeq & \pi_1(G) \\ \downarrow p^* & & \downarrow \text{multiplication by } k \\ H^2(\hat{F}; \pi_1(G)) & \simeq & \pi_1(G) \end{array}$$

Since $\pi_1(G)$ is free abelian, it follows that $o_2(p^*\xi)$ is nonzero. Hence ξ is not virtually trivial. Q.E.D.

REMARKS. (1) For an application of obstruction theory for the (somewhat analogous) case of G locally isomorphic to $PSL(2, R)$ see [1, §3]. The first examples of flat G -bundles which are not virtually trivial are due to Milnor [5], where $G = SL(2; R)$.

(2) If G is a connected solvable Lie group and G' is simply connected, then in general it is necessary to pass to a finite covering to trivialize a flat principal G -bundle. Here is a simple example. Let $M = RP^2$ and $G = SO(2)$. There are two equivalence classes of flat principal G -bundles over M : the trivial bundle with total space $RP^2 \times SO(2)$, and the bundle ξ induced by a faithful representation $Z/2 \rightarrow SO(2)$. The total space of ξ is easily seen to be homeomorphic to $S^2 \times SO(2)$. Clearly these two spaces are distinct, so that ξ is nontrivial.

However, the above proof shows that if $H_1(M; Z)$ is free abelian, then every flat principal G -bundle over M is trivial.

(3) In Sullivan [6] it is shown that if G is an algebraic group, then there are only a finite number of isomorphism classes of principal G -bundles over a base space M as above which admit flat structures. For some G not algebraic counterexamples may be constructed along the lines of (b) $\Rightarrow$ (a) above. Here is the simplest example. Let $\tilde{G}$ be the Heisenberg group consisting of real upper unitriangular 3×3 matrices, and let C be the cyclic group generated by any nontrivial $c \in \text{center}(\tilde{G}) = (\tilde{G})'$. Let $G = \tilde{G}/C$. For every $n \in Z$ there exist $\tilde{a}, \tilde{b} \in \tilde{G}$ with $[\tilde{a}, \tilde{b}] = c^n$. The images of $\tilde{a}, \tilde{b}$ in G commute so they determine a homomorphism $Z + Z \rightarrow G$. The associated flat principal G -bundle over the 2-torus is fiber homotopy equivalent to a circle bundle with Euler number n . Thus infinitely many principal bundles occur.

We may interpret these examples in terms of the theory developed by Gromov [3]. In [3] it is shown that for algebraic groups G , the image of the natural homomorphism $H^*(BG) \rightarrow H^*(BG^\delta)$ consists of bounded cohomology classes. (Here G^δ denotes G with the discrete topology.) It follows from the preceding examples that for some nilpotent nonalgebraic Lie groups G , the image of $H^*(BG) \rightarrow H^*(BG^\delta)$ consists entirely of *unbounded* cohomology classes.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139