A SIMPLE PROOF THAT THE CONCORDANCE GROUP OF ALGEBRAICALLY SLICE KNOTS IS INFINITELY GENERATED¹

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ABSTRACT. A simple proof of the result stated in the title is obtained by making the Casson-Gordon invariant additive.

A. J. Casson and C. McA. Gordon proved in [1], [2] that there are algebraically slice knots which are not slice knots. In other words, the concordance group $\mathcal{Q}$ of algebraically slice knots is nontrivial. A natural question: Is $\mathcal{Q}$ infinitely generated? The author learned from Wu-chung Hsiang that Casson had obtained the affirmative answer to this question for some time; however, Casson has not published his proof to date. The purpose of this short note is to present a simple proof, which would possibly be different from Casson's. The main point in our argument is the observation that we can make the Casson-Gordon invariant additive by slightly generalizing its definition, thereby making it possible to detect linear independence in $\mathcal{Q}$. We shall use the language and notation of [2].

1. Additivity of Casson-Gordon invariant for 3-manifolds. In [2], Casson and Gordon defined, for a closed oriented 3-manifold M and an epimorphism $\phi: H_1(M) \rightarrow \mathbb{Z}_m$, the invariant $\sigma_r(M, \phi)$, $0 < r < m$. The definition goes as follows. Suppose $\tilde{M} \rightarrow M$ is the m -fold cyclic covering induced by ϕ . Pick up an m -fold cyclic branched covering of 4-manifolds $\tilde{W} \rightarrow W$, branched over a surface $F \subset \text{int } W$, such that $\partial(\tilde{W} \rightarrow W) = (\tilde{M} \rightarrow M)$. (The existence of such (W, F) follows from Lemma 2.2 of [2].) Then define

$$\sigma_r(M, \phi) = \text{sign } W - \epsilon_r(\tilde{W}) - \frac{2[F]^2 r(m-r)}{m^2}.$$

For our purpose, we have to deal with arbitrary homomorphism ϕ . For simplicity, we shall restrict ourselves to the case $m = p$, a prime, so that a homomorphism $H_1(M) \rightarrow \mathbb{Z}_p$ is either epimorphic or trivial.

DEFINITION. Let $\phi: H_1(M) \rightarrow \mathbb{Z}_p$ be a homomorphism, where M is an oriented closed 3-manifold, p a prime. Define $\sigma_r(M, \phi)$ as above if ϕ is epimorphic, and define $\sigma_r(M, \phi) = 0$ if ϕ is trivial, for $0 < r < p$.

This invariant is additive in the following sense. Let M', M'' be two closed

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oriented 3-manifolds. We know $H_1(M' \# M'') = H_1(M') \oplus H_1(M'')$, so that every pair of homomorphisms $\phi': H_1(M') \rightarrow \mathbb{Z}_p$, $\phi'': H_1(M'') \rightarrow \mathbb{Z}_p$ determines uniquely a homomorphism $\phi = \phi' \oplus \phi'': H_1(M' \# M'') \rightarrow \mathbb{Z}_p$, and vice versa.

LEMMA 1. $\sigma_r(M' \# M'', \phi' \oplus \phi'') = \sigma_r(M', \phi') + \sigma_r(M'', \phi'')$, $0 < r < p$.

PROOF. *Case 1.* Both of ϕ' , ϕ'' are epimorphic. Let (W', F') be the data needed for defining $\sigma_r(M', \phi')$, and (W'', F'') be those for $\sigma_r(M'', \phi'')$. Take $(W, M) = (W', M') \# (W'', M'')$ and $F = F' \cup F''$. Then we may use (W, F) for defining $\sigma_r(M' \# M'', \phi' \oplus \phi'')$. Now W is obtained by pasting W' and W'' together along a 3-disk, and $\tilde{W}$ is obtained by pasting $\tilde{W}'$ and $\tilde{W}''$ together along p 3-disks, neither of which intersects the branching set F . By the Mayer-Vietoris sequence, we see that the intersection form on $H_2(W)$ is the orthogonal sum of those on $H_2(W')$ and $H_2(W'')$ (hence $\text{sign } W = \text{sign } W' + \text{sign } W''$ and $[F]^2 = [F']^2 + [F'']^2$), and the intersection form on $H_2(\tilde{W})$ is the $\mathbb{Z}_p$ -equivariant orthogonal sum of $H_2(\tilde{W}')$ and $H_2(\tilde{W}'')$ (hence $\varepsilon_r(\tilde{W}) = \varepsilon_r(\tilde{W}') + \varepsilon_r(\tilde{W}'')$). Therefore

$$\sigma_r(M' \# M'', \phi' \oplus \phi'') = \sigma_r(M', \phi') + \sigma_r(M'', \phi''), \quad 0 < r < p.$$

Case 2. ϕ' is epimorphic but ϕ'' is trivial. Let (W', F') be the data needed for defining $\sigma_r(M', \phi')$. Take any W'' such that $\partial W'' = M''$ and $H_1(W'') = 0$. Take $(W, M) = (W', M') \# (W'', M'')$ and $F = F'$. Then we may use (W, F) for defining $\sigma_r(M' \# M'', \phi' \oplus \phi'')$. The same argument as in Case 1 still works if we take F'' to be empty and take $\tilde{W}''$ to be the disjoint union of p copies of W'' , with the obvious $\mathbb{Z}_p$ action by cyclic permutation. So we get $\text{sign } W = \text{sign } W' + \text{sign } W''$, $\varepsilon_r(\tilde{W}) = \varepsilon_r(\tilde{W}') + \varepsilon_r(\tilde{W}'')$, $[F]^2 = [F']^2$. But for this $\mathbb{Z}_p$ -action on $\tilde{W}''$ it is easily seen that $\varepsilon_r(\tilde{W}'') = \text{sign } W''$ for all $0 < r < p$. Therefore

$$\sigma_r(M' \# M'', \phi \oplus \phi'') = \sigma_r(M', \phi') = \sigma_r(M', \phi') + \sigma_r(M'', \phi''), \quad 0 < r < p.$$

Case 3. Both of ϕ' , ϕ'' are trivial. This case is trivial.

2. Additivity of Casson-Gordon invariant for knots. Let K be a knot in S^3 , $M_n(K)$ be the 2^n -fold branched covering of (S^3, K) , $\phi: H_1(M_1(K)) \rightarrow \mathbb{Z}_m$ be a homomorphism. By composition with the surjection induced by branched covering projection $M_n(K) \rightarrow M_1(K)$, ϕ determines $\phi_n: H_1(M_n(K)) \rightarrow \mathbb{Z}_m$. The Casson-Gordon invariant for (K, ϕ) is $\sigma_r(M_n(K), \phi_n)$, $0 < r < m$. It was originally defined in [2] for epimorphic ϕ , but now it also makes sense for arbitrary ϕ when $m = p$.

This invariant is additive in the following sense. Let K' , K'' be two oriented knots in S^3 . Then $M_n(K' \# K'') = M_n(K') \# M_n(K'')$, so that every pair of homomorphisms $\phi': M_1(K') \rightarrow \mathbb{Z}_p$ and $\phi'': M_1(K'') \rightarrow \mathbb{Z}_p$ determines a unique $\phi = \phi' \oplus \phi'': H_1(M_1(K' \# K'')) \rightarrow \mathbb{Z}_p$, and $\phi_n = \phi'_n \oplus \phi''_n$. Now, a direct consequence of Lemma 1 is

LEMMA 2. $\sigma_r(M_n(K' \# K''), \phi'_n \oplus \phi''_n) = \sigma_r(M_n(K'), \phi'_n) + \sigma_r(M_n(K''), \phi''_n)$, for $0 < r < p$.

3. Doubled knots of the trivial knot. Let us quote from [2] some results about doubled knots. Let K_k be the k -twisted double of the unknot, as depicted on [2, p. 46]. K_k is known to be algebraically slice iff $4k + 1 = l^2$ for some integer l . Let us

rewrite K_k as $K^{(l)}$ if $4k + 1 = l^2$. Then $K^{(l)}$, l taking odd values, are algebraically slice knots and represent elements of the concordance group $\mathcal{Q}$. $K^{(l)}$ is a slice knot iff $l = 1$ or 3 . The computation in [2, §5] can be summarized as

LEMMA 3. Let $\bar{\mu}_1$ be the generator of $H_1(M_1(K^{(l)})) (\cong \mathbb{Z}_{l^2})$ specified on [2, p. 48]. Suppose an epimorphism $\phi: H_1(M_1(K^{(l)})) \rightarrow \mathbb{Z}_m$ sends $\bar{\mu}_1$ to $q \in \mathbb{Z}_m$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \sigma_r(M_n(K^{(l)}), \phi_n) = 2 \left\{ 1 + \left[\frac{(l^2 - 1)r'}{2m} \right] - r'(m - r') \left(\frac{l}{m} \right)^2 \right\},$$

for $0 < r < m$,

where $0 < r' \leq (m - 1)/2$ satisfies $r' \equiv \pm qr \pmod{m}$. If $l > 5$, the right-hand side is always negative.

In fact, the computation in [2] is carried out for $q = 1$. But we can use the following general fact which can be easily proved by means of Novikov additivity: For a closed 3-manifold M and two epimorphisms $\phi, \phi': H_1(M) \rightarrow \mathbb{Z}_m$, related by $\phi = q\phi'$ where q is coprime to m , we have $\sigma_r(M, \phi) = \sigma_r(M, \phi')$, $0 < r < m$, where $r' \equiv qr \pmod{m}$ and $0 < r' < m$.

4. Infinite-generatedness of $\mathcal{Q}$.

THEOREM. Let P be the set of prime numbers ≥ 5 . Then, the set $\{K^{(p)}\}_{p \in P}$ is linearly independent in $\mathcal{Q}$.

For a proof, let us consider a knot

$$K = k_1 K^{(p_1)} \# \dots \# k_t K^{(p_t)},$$

where $p_1, \dots, p_t \in P$, $p_i \neq p_j$ for $i \neq j$, $k_1, \dots, k_t$ are nonzero integers. We want to prove that K is not slice. But in view of Theorem 4.1 of [2], it suffices to prove the following.

LEMMA 4. (1) For any subgroup G of $H_1(M_1(K))$ with $|G|^2 = |H_1(M_1(K))|$, there exists an epimorphism $\phi: H_1(M_1(K)) \rightarrow \mathbb{Z}_{p_1}$ satisfying $\phi(G) = 0$.

(2) For any epimorphism $\phi: H_1(M_1(K)) \rightarrow \mathbb{Z}_{p_1}$,

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \sigma_r(M_n(K), \phi_n) < 0 \quad \text{if } k_1 > 0,$$

$$> 0 \quad \text{if } k_1 < 0.$$

PROOF. (1) The factor group $H_1(M_1(K))/G$ has order $|G| = p_1^{|k_1|} \dots p_t^{|k_t|}$, hence it has $\mathbb{Z}_{p_1}$ as a factor group. So there exists an epimorphism $\phi: H_1(M_1(K)) \rightarrow \mathbb{Z}_{p_1}$.

(2) For short, let us write M_n for $M_n(K)$, $k_i M_n^{(i)}$ for $M_n(k_i K^{(p_i)})$, $K^{(i,j)}$ for the j th copy of $\pm K^{(p_i)}$ in $k_i K^{(p_i)}$, $M_n^{(i,j)}$ for $M_n(K^{(i,j)})$, $1 \leq j \leq |k_i|$, $1 \leq i \leq t$. Then

$$M_n = k_1 M_n^{(1)} \# \dots \# k_t M_n^{(t)},$$

$$H_1(M_1) = H_1(k_1 M_n^{(1)}) \oplus \dots \oplus H_1(k_t M_n^{(t)})$$

$$\cong (\mathbb{Z}_{p_1}^{|k_1|}) \oplus \dots \oplus (\mathbb{Z}_{p_t}^{|k_t|}).$$

Recall that $-K$ means the mirror image of K , changing the sign of a knot also changes the sign of its Casson-Gordon invariant. Hence we only have to consider the $k_1 > 0$ case.

Write $\phi^{(i)} = \phi|_{H_1(k_i M_1^{(i)})}$, $\phi^{(i,j)} = \phi|_{H_1(M^{(i,j)})}$. Then, for $i > 1$, $\phi^{(i)}$ is trivial because p_i is coprime to p_1 . By Lemma 2,

$$\sigma_r(M_n, \phi_n) = \sigma_r(k_1 M_n^{(1)}, \phi_n^{(1)}) = \sum_{j=1}^{k_1} \sigma_r(M_n^{(1,j)}, \phi_n^{(1,j)}).$$

But ϕ is epimorphic, so that at least one of $\phi^{(1,j)}$, $1 \leq j \leq k_1$, will be epimorphic. The conclusion of the lemma then follows from Lemma 3.

The proof of the Theorem is now complete.

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