ON THE DIMENSION OF μ -SPACES

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ABSTRACT. The main results are as follows:

THEOREM 1. Let $\mathcal{C}$ be the class of all μ -spaces. Then the following are equivalent for every $X \in \mathcal{C}$: (1) $\dim X < n$, (2) there exists a closed mapping f of $Z \in \mathcal{C}$ with $\dim Z < 0$ onto X such that $\text{ord } f < n + 1$, (3) $X = \bigcup_{i=1}^{n+1} X_i$, where $\dim X_i < n$ for each i and (4) $\text{Ind } X < n$.

THEOREM 2. A space X is a μ -space with $\dim X < n$ if and only if X is the inverse limit of an inverse sequence $\{X_i, g_i^j\}$ of paracompact σ -metric spaces X_i such that $\dim X_i < n$ for every $i \in N$.

As the applications of them, some product theorems for covering dimension are given.

Introduction. In the sequel, all spaces are assumed to be Hausdorff topological spaces, mappings to be continuous onto and N to denote the positive integers. In this paper, we consider the following fundamental theorems of dimension for a class $\mathcal{C}$ of topological spaces X : (1) $\dim X < n$, (2) there exists a closed mapping f of $Z \in \mathcal{C}$ with $\dim Z < 0$ onto X such that $\text{ord } f < n + 1$, (3) $X = \bigcup_{i=1}^{n+1} X_i$, where $\dim X_i < 0$ for each i and (4) $\text{Ind } X < n$. As is well known, the statements (1)–(4) are equivalent for the class of metric spaces. Nagami defined in [7] the class of paracompact σ -metric spaces and proved that they are equivalent for this class. Moreover, in [4] he also proved that for every cubic μ -space X (1), (3), (4) and (2) with replaced Z such that Z is a μ -space are equivalent. Recently in [5] he defined the class $\mathcal{C}$ of free L -spaces which is an intermediate one between those of Lašnev and M_1 -spaces, and proved that they are equivalent for this class. A free L -space is shown there to be a μ -space, but not the converse. The aim of this paper is to prove that (1)–(4) are equivalent for the class of all μ -spaces and to give some variations.

DEFINITION 1 [7, Definition 1]. A space X is called a σ -metric space if $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is a closed metrizable subspace of X , and $\{X_i\}$ is called a *scale* of X . A scale $\{X_i\}$ is called *monotone* if $X_i \subset X_{i+1}$ for every $i \in N$.

In the sequel, every scale is assumed to be monotone because X has a scale if and only if X has a monotone scale. The following argument is due to Nagami [7]: If X is a paracompact σ -metric space with a scale $\{X_i\}$, then there exists a contraction, that is, a one-to-one mapping ρ of X onto a metric space $\hat{X}$ such that $\rho|_{X_m}$ is a homeomorphism of X_m onto a closed subspace $\rho(X_m)$ of $\hat{X}$. Such a pair $(\hat{X}, \rho)$ is called a *replica* of X with respect to $\{X_i: i \in N\}$.

Received by the editors June 30, 1980 and, in revised form, November 18, 1980.

1980 *Mathematics Subject Classification*. Primary 54F45.

Key words and phrases. σ -metric space, μ -space, cubic μ -space, inverse limit.

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0002-9939/81/0000-0444/\$02.50

DEFINITION 2. A space X is called a μ -space if X is embedded in the countable product of paracompact σ -metric spaces. Especially if X is the countable product itself of paracompact σ -metric spaces, then X is called a *cubic μ -space*.

We observe that every μ -space is paracompact and perfectly normal, and this fact is used in the discussion below. Every paracompact σ -metric space is obviously a σ -space, that is, a space with a σ -locally finite network in the sense of Okuyama [8]. Since paracompact σ -spaces are countably productive [8, Theorem 4.7], every cubic μ -space is also a paracompact σ -space. Moreover, since by [8, Theorems 2.1 and 2.9] paracompact σ -spaces are hereditary with respect to any subspace, so is every μ -space. Thus by [8, Theorems 2.8 and 2.9] every μ -space is paracompact and perfectly normal.

Every free L -space is a μ -space [5, Corollary 3.5]. But the converse is not true. In fact let X be the butterfly space which is first described by McAuley. As Heath proved, X is not an M_1 -space, though it is a paracompact σ -metric space. On the other hand by the embedding theorem of free L -spaces [5, Theorem 3.4] and [3, Theorem 2] every free L -space is hereditarily M_1 . Therefore X is not a free L -space.

LEMMA 1 (PASYNKOV'S FACTORIZATION THEOREM [1, 4.2G]). *Let $f: X \rightarrow Y$ be a mapping of a normal space X onto a metrizable space Y . Then there exists a metrizable space Z and mappings $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $\dim Z < \dim X$ and $f = hg$.*

LEMMA 2 [7, THEOREM 6]. *Let X be a paracompact σ -metric space with its replica $(\hat{X}, \rho)$, let W be a metric space and $g: W \rightarrow \hat{X}$ be a mapping. Construct the subspace Z of $X \times W$ as follows:*

$$Z = \{(x, y) \in X \times W: \rho(x) = g(y)\}.$$

Let $f: Z \rightarrow X$, $\sigma: Z \rightarrow W$ be the restrictions to Z of the projections. Then Z is a paracompact σ -metric space with its replica (W, σ) and $\rho f = g\sigma$. If, moreover, g is a closed mapping with $\text{ord } g \leq n + 1$ (or a one-to-one mapping) then f is a closed mapping with $\text{ord } f \leq n + 1$ (or a one-to-one mapping), respectively.

LEMMA 3 (FACTORIZATION THEOREM FOR PARACOMPACT σ -METRIC SPACES). *Let $f: X \rightarrow Y$ be a mapping of a normal space X onto a paracompact σ -metric space Y . Then there exist a paracompact σ -metric space Z and mappings $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $\dim Z < \dim X$ and $f = hg$.*

PROOF. Let (Y', ρ) be the replica of Y . By Lemma 1, there exist a metric space Z' with $\dim Z' \leq \dim X$ and mappings $g': X \rightarrow Z'$ and $h': Z' \rightarrow Y'$ such that $\rho f = h'g'$. Construct the subspace Z of $Y \times Z'$ as follows:

$$Z = \{(x, y) \in Y \times Z': \rho(y) = h'(x)\}.$$

Let $h: Z \rightarrow Y$ and $\sigma: Z \rightarrow Z'$ be the restrictions to Z of the projections. Then by Lemma 2, (Z', σ) is the replica of a paracompact σ -metric space Z and therefore by the countable sum theorem $\dim Z < \dim Z' \leq \dim X$. Let $g: X \rightarrow Y \times Z'$ be defined by $g(x) = (f(x), g'(x))$ for every $x \in X$. Then $Z = g(X)$ and g is a mapping such that $f = hg$.

LEMMA 4 [4, LEMMA 3]. Let $X_i, i \in N$, be paracompact σ -metric spaces with $\dim X_i = 0$. Then $\dim \prod_{i=1}^{\infty} X_i = 0$.

THEOREM 1. Let $\mathcal{C}$ be the class of all μ -spaces. Then the following are equivalent for $X \in \mathcal{C}$.

- (1) $\dim X \leq n$.
- (2) There exists a closed mapping f of $Z \in \mathcal{C}$ with $\dim Z < 0$ onto X such that $\text{ord } f \leq n + 1$.
- (3) $X = \bigcup_{i=1}^{n+1} X_i$, where $\dim X_i \leq 0$ for every i .
- (4) $\text{Ind } X \leq n$.

PROOF. (1) $\rightarrow$ (2). Part 1. Let $X \subset \prod_{i=1}^{\infty} X_i$ and $\dim X \leq n$, where each X_i is a paracompact σ -metric space. For each $i \in N$, let (X'_i, ρ_i) be the replica of X_i . For each $i \in N$, set

$$\begin{aligned} g_1 &= (\rho_1 \times \rho_2 \times \cdots)|X, \quad Y_1 = g_1(X), \\ g_i &= (\text{id}_{X_1} \times \cdots \times \text{id}_{X_{i-1}} \times \rho_i \times \rho_{i+1} \times \cdots)|X, \\ Y_i &= g_i(X) \quad \text{for } i > 1. \end{aligned}$$

For each pair $i > j$, set

$$g_j^i = (\text{id}_{X_1} \times \cdots \times \text{id}_{X_{j-1}} \times \rho_j \times \cdots \times \rho_{i-1} \times \text{id}_{X_i} \times \text{id}_{X_{i+1}} \times \cdots)|Y_i.$$

Then $g_j^i: Y_i \rightarrow Y_j$ is a one-to-one mapping. Especially (Y_1, g_1^i) is a replica of a paracompact σ -metric space Y_i . Obviously $X = \varprojlim \{Y_i, g_j^i\}$. By Lemma 1 there exist a metric space M_1 with $\dim M_1 \leq n$ and mappings $h_1: X \rightarrow M_1$ and $k_1: M_1 \rightarrow Y_1$ such that $g_1 = k_1 h_1$. Since g_1 is one-to-one and h_1 is onto, both of h_1 and k_1 are chosen to be one-to-one. By Lemma 3 and its proof there exist a paracompact σ -metric space M_i with $\dim M_i \leq n$ and one-to-one mappings $h_i: X \rightarrow M_i$, $k_i: M_i \rightarrow Y_i$ and $\sigma_1^i: M_i \rightarrow M_1$ such that $g_i = k_i h_i$ and (M_1, σ_1^i) is a replica of M_i . For each pair $i > j$, define $\sigma_j^i: M_i \rightarrow M_j$ by $\sigma_j^i = (\sigma_1^i)^{-1} \sigma_1^j$. Since by the proof of Lemma 3, M_j has the open base

$$\{(\sigma_1^i)^{-1}(U) \cap (k_j)^{-1}(V): U, V \text{ are open in } M_1, Y_j, \text{ respectively}\},$$

σ_j^i is continuous. If we define a mapping $k: M = \varprojlim \{M_i, \sigma_j^i\} \rightarrow X$ by $k(\langle m_i \rangle) = \langle k_i(m_i) \rangle$, $\langle m_i \rangle \in M$, then k is a homeomorphism of M onto X . Thus we have $X = \varprojlim \{M_i, \sigma_j^i\}$.

Part 2. Since M_1 is a metric space with $\dim M_1 \leq n$, there exists a closed mapping f_1 of a metric space Z_1 onto M_1 such that $\dim Z_1 < 0$ and $\text{ord } f \leq n + 1$. By Lemma 2 again, for each i there exists a closed mapping f_i of a paracompact σ -metric space Z_i with $\dim Z_i < 0$ onto M_i such that $\text{ord } f_i \leq n + 1$, and there exists a contraction $\lambda_1^i: Z_i \rightarrow Z_1$ such that (Z_1, λ_1^i) is a replica of Z_i and $\sigma_1^i f_i = f_1 \lambda_1^i$. For each pair $i > j$, $\lambda_j^i: Z_i \rightarrow Z_j$ is defined similarly to the above case σ_j^i . Set $Z = \varprojlim \{Z_i, \lambda_j^i\}$ and define a mapping $f: Z \rightarrow X$ such that $h_1 f = f_1 \lambda_1$, where $\lambda_i: Z \rightarrow Z_i$ is the restriction of the projection. As shown in the proof of [4, Theorem 2], f is a closed mapping of a μ -space Z onto X such that $\text{ord } f \leq n + 1$. From

Lemma 4, $\dim Z \leq 0$ follows. (2) $\rightarrow$ (3): If we set $X_i = \{x \in X: |f^{-1}(x)| = i\}$, then $X = \bigcup_{i=1}^{n+1} X_i$ and $\dim X_i \leq 0$ by [4, Lemma 4]. (3) $\rightarrow$ (4) and (4) $\rightarrow$ (1) are well-known results that hold in the classes of completely normal and normal spaces, respectively. Thus the proof is completed.

LEMMA 5. Let X, Y be paracompact σ -metric spaces with replicas $(X', \rho_1), (Y', \rho_2)$, respectively, and let $g: Y \rightarrow X$ be a mapping. Then there exist a metric space Z and mappings $\rho: Y \rightarrow Z$ and $g': Z \rightarrow X'$ such that (Z, ρ) is a replica of Y and $\rho_1 g = g' \rho$.

PROOF. $\rho: Y \rightarrow Y' \times X'$ is defined by

$$\rho(y) = (\rho_2(y), \rho_1(g(y))), \quad y \in Y.$$

Then $Z = \rho(Y)$ is a metric space because X', Y' are so. Since ρ_2 is one-to-one, so is ρ . Let $\{Y_m: m \in N\}$ be the scale of Y with respect to which (Y', ρ_2) is the replica. Since $\rho_2|Y_m: Y_m \rightarrow \rho_2(Y_m)$ is a homeomorphism onto a closed subspace of Y' , $\rho|Y_m: Y_m \rightarrow \rho(Y_m)$ is a homeomorphism onto a closed subspace of Z . Therefore (Z, ρ) is the replica of Y . Let g' be the restriction to Z of the projection. Then g' is the desired mapping.

LEMMA 6 (THEOREM ON DIMENSION OF THE LIMIT OF AN INVERSE SEQUENCE [1, THEOREM 4.1.22]). If X is the limit of an inverse sequence $\{X_i, \pi_j^i\}$ of metric spaces such that $\dim X_i \leq n$ for every $i \in N$, then $\dim X \leq n$.

THEOREM 2. A space X is a μ -space with $\dim X \leq n$ if and only if X is the limit of an inverse sequence $\{X_i, g_j^i\}$ of paracompact σ -metric spaces X_i such that $\dim X_i \leq n$ for every $i \in N$.

PROOF. The "only if" part is proved in Part I of the proof of Theorem 1. The "if" part: Suppose that we are given an inverse sequence $\{X_i, g_j^i\}$ consisting of paracompact σ -metric spaces X_i such that $\dim X_i \leq n$ for every $i \in N$ and $X = \lim \{X_i, g_j^i\}$. Let (M_1, ρ_1) be the replica of X_1 with respect to the scale $\{X_{1m}: m \in \tilde{N}\}$. By repeated use of Lemma 5, we have an inverse limit sequence $\{M_i, h_j^i\}$ and a sequence $\{\rho_i: i \in N\}$ of contractions such that (M_i, ρ_i) is the replica of X_i and $\rho_i g_i^{i+1} = h_i^{i+1} \rho_{i+1}$ for each $i \in N$. Set

$$g_i = (\text{id}_{X_1} \times \cdots \times \text{id}_{X_{i-1}} \times \rho_i \times \rho_{i+1} \times \cdots)|X,$$

$$Y_i = g_i(X) \subset \prod_{j < i} X_j \times \prod_{j > i} M_j.$$

$$\begin{array}{ccccccc} X_1 & \xleftarrow{g_1^2} & X_2 & \xleftarrow{g_2^3} & X_3 & \xleftarrow{g_3^4} & \\ \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \\ M_1 & \xleftarrow{h_1^2} & M_2 & \xleftarrow{h_2^3} & M_3 & \xleftarrow{h_3^4} & \end{array}$$

For each pair $i > j$, set

$$k_j^i = (\text{id}_{X_1} \times \cdots \times \text{id}_{X_{j-1}} \times \rho_j \times \cdots \times \rho_{i-1} \times \text{id}_{M_i} \times \cdots)|Y_i.$$

Then $k_j^i: Y_i \rightarrow Y_j$ is a one-to-one mapping. Obviously $X = \lim \{Y_i, k_j^i\}$ and Y_1 is a metric space such that each $Y_i, i > 1$, has the replica (Y_1, k_1^i) . We shall show that

$\dim Y_i < n$ for every $i \in N$. Note that $Y_1 = \lim \{M_i, h_j^i\}$. Since each M_i is a metric space with $\dim M_i < n$, by Lemma 6 we have $\dim Y_1 < n$. Let $i > 2$. Let $\{X_{im}: m \in N\}$ be a scale of X_i with respect to which (M_i, ρ_i) is the replica. Set

$$\Lambda = N^{i-1} = \{(m_1, \dots, m_{i-1}): m_1, \dots, m_{i-1} \in N\}.$$

Then Λ is a countable set. For each $\lambda = (m_1, \dots, m_{i-1}) \in \Lambda$. Set

$$Y_i(\lambda) = Y_i \cap \left(\prod_{j=1}^{i-1} X_{jm_j} \times \prod_{j=i}^{\infty} M_j \right).$$

Since $\rho_j|_{X_{jm_j}}$ is a homeomorphism of X_{jm_j} onto a closed subspace $\rho_j(X_{jm_j}) = M_{jm_j}$, $Y_i(\lambda)$ is homeomorphic to the inverse limit $\lim \{M_i, h_j^i\}$. Since $\dim \lim \{M_i, h_j^i\} < n$, $\dim Y_i(\lambda) < n$. Since $Y_i(\lambda)$ is closed in Y_i , by the countable sum theorem $\dim Y_i < n$. By the second part of the proof of Theorem 1 it is similarly shown that X is the image of a μ -space Z with $\dim Z < 0$ by the closed mapping f such that $\text{ord } f < n + 1$. Thus by Theorem 1, (1) $\leftrightarrow$ (2), we have $\dim X < n$.

COROLLARY 1. *If X is paracompact and perfectly normal, Y is a μ -space and at least one of them is nonempty, then*

$$\dim(X \times Y) < \dim X + \dim Y.$$

PROOF. Suppose $\dim X < m$ and $\dim Y < n$. By Theorem 2, $Y = \lim \{Y_i, h_j^i\}$, where each Y_j is a paracompact σ -metric space such that $\dim Y_j < m$. Then obviously $X \times Y$ is the inverse limit of $\{X \times Y_j, g_j^i\}$, where each $g_j^i = \text{id}_X \times h_j^i$. From the result of [8, Theorem 4.6] that the product of paracompact perfectly normal space and paracompact σ -space is perfectly normal, it follows that each $X \times Y_j$ is perfectly normal. Observe that $\dim(X \times Y_j) < m + n$ by [9, Corollary 3.6, p. 354] and the countable sum theorem. Therefore by [2, Proposition 2] we have $\dim(X \times Y) < m + n$.

COROLLARY 2. *If X, Y are μ -spaces of which at least one is nonempty, then $X \times Y$ is a μ -space such that*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

With respect to the Cartesian product theorem for covering dimension, there exists the following:

LEMMA 7 [9, PROPOSITION 3.5, p. 354]. *If Y is a metrizable space and X is a space such that $X \times Y$ is countably paracompact and normal and at least one of them is nonempty, then*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Finally, we state the Cartesian product theorem for cubic μ -spaces. Of course every metric space is a cubic μ -space. But the converse is not true, because by [7, Theorem 1] the countable product of nonmetrizable σ -metric spaces is not σ -metric.

THEOREM 3. *If Y is a cubic μ -space and X is a space such that $X \times Y$ is countably paracompact and normal and at least one of them is nonempty, then*

$$\dim(X \times Y) < \dim X + \dim Y.$$

PROOF. Let $Y = \prod_{i=1}^{\infty} Y_i$, where each Y_i is a paracompact σ -metric space. For each $i \in N$ let $P_i = \prod_{j < i} Y_j$ and let $g_i^{i+1}: P_{i+1} \rightarrow P_i$ be the projection. Then $X \times Y$ is the inverse limit of $\{X \times P_i, \text{id}_X \times g_i^{i+1}\}$, where $\text{id}_X \times g_i^{i+1}$ is an open mapping. Note that each $X \times P_i$ is countably paracompact and normal and therefore by Lemma 7 and by the countable sum theorem $\dim(X \times P_i) \leq \dim X + \dim Y$. By [6, Theorem 1.7] we have $\dim(X \times Y) \leq \dim X + \dim Y$.

REFERENCES

1. R. Engelking, *Dimension theory*, North-Holland, Amsterdam, 1978.
2. M. G. Charalambous, *The dimension of inverse limits*, Proc. Amer. Math. Soc. **58** (1976), 289–295.
3. T. Mizokami, *On Nagata's problem for paracompact σ -metric spaces*, Topology Appl. **11** (1980), 211–221.
4. K. Nagami, *Perfect class of spaces*, Proc. Japan Acad. **48** (1972), 21–24.
5. ———, *Dimension of free L-spaces*, Fund. Math. **108** (1980), 211–224.
6. ———, *Countable paracompactness of inverse limits and products*, Fund. Math. **73** (1971/72), 261–270.
7. ———, *Dimension for σ -metric spaces*, J. Math. Soc. Japan **23** (1971), 123–129.
8. A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **9** (1967), 236–254.
9. A. R. Pears, *Dimension theory for general spaces*, Cambridge Univ. Press, New York, 1975.

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