ON SPLITTING AUGMENTATION IDEALS

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ABSTRACT. Let G be a group, H a subgroup of G and R an associative ring. Write $\omega(RG)$ for the augmentation ideal of the group ring RG , and $\omega(RH)G$ for the right ideal of RG generated by $\omega(RH)$. For G finitely generated over H we characterize, in terms of the Bass-Serre theory of groups acting on trees, the situation where $\omega(RH)G$ is an RG -summand of $\omega(RG)$.

Let G be a group, H a subgroup of G and R an associative ring with $1 \neq 0$.

We write RG for the group ring, $\omega(RG)$ for its augmentation ideal, and $\omega(RH)G$ for the right ideal of RG generated by $\omega(RH)$.

The situation where $\omega(RH)G$ is an RG -summand of $\omega(RG)$ has been considered by various authors [1], [4], [5], [6], and the purpose of this paper is to give the following characterization in terms of the Bass-Serre theory of groups acting on trees. (We write R^{-1} for the set of positive integers invertible in R ; for other terminology we refer to [2].)

THEOREM A. *If G is finitely generated over H then the following are equivalent:*

- (1) $\omega(RH)G$ is an RG -summand of $\omega(RG)$.
- (2) G can be expressed as the fundamental group of a graph of groups $(\mathcal{G}, Y)$ with Y finite, with finite edge groups, with H as one of the vertex groups, such that for each infinite vertex group $\mathcal{G}(v)$ distinct from H , $|H \cap \mathcal{G}(v)^g| \in R^{-1}$ for all $g \in G$, and with $|H \cap H^g| \in R^{-1}$ for all $g \notin H$.
- (3) G acts on a tree X with X/G finite, with finite edge stabilizers, with a vertex v_0 having stabilizer precisely H , such that for each $v \neq v_0$ having star (v) infinite $|H \cap G_v| \in R^{-1}$, and with $|H \cap H^g| \in R^{-1}$ for all $g \notin H$.

§1 below is devoted to proving this. The equivalence of (2) and (3) is immediate from the Bass-Serre Structure Theorem; the equivalence of (1) and (2) will be derived using a result proved in [3] and formulated in [2, p. 81] as follows.

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DUNWOODY'S THEOREM. *Let $d: G \rightarrow RG$ be a derivation with kernel H . If G is finitely generated over H then G acts on a tree X with X/G finite, with finite edge stabilizers, with a vertex v_0 having stabilizer precisely H , such that there exists a right RG -linear map $\beta: R[E(X)] \rightarrow RG$ such that $X(v_0, v_0g)\beta = (g)d$ for all $g \in G$.*

§2 below illustrates Theorem A with examples and applications to cases previously considered in the literature.

1. Proof of Theorem A. Let us restate the condition

(1) $\omega(RH)G$ is an RG -summand of $\omega(RG)$

as

(4) There is a right RG -linear map $\alpha: \omega(RG) \rightarrow \omega(RH)G$ such that $(1 - h)\alpha = (1 - h)$ for all $h \in H$.

Writing $\pi: RG \rightarrow R[H \setminus G]$, $g \mapsto Hg$, we claim that (4) is equivalent to

(5) There is a derivation $d: G \rightarrow RG$ such that $(g)d\pi = H(1 - g)$ for all $g \in G$, and $(h)d = 0$ for all $h \in H$.

PROOF. (4) $\Rightarrow$ (5). Define $(g)d = (1 - g) - (1 - g)\alpha$.

(5) $\Rightarrow$ (4). Define $(1 - g)\alpha = (1 - g) - (g)d$ and notice that the kernel of π is $\omega(RH)G$, so α has the desired codomain.

Now the kernel of d in (5) is precisely H so assuming G is finitely generated over H we can apply Dunwoody's Theorem to deduce that (5) implies

(6) G acts on a tree X
with X/G finite,
with finite edge stabilizers,
with a vertex v_0 having stabilizer precisely H ,
such that there exists a right RG -linear map $\beta: R[E(X)] \rightarrow RG$ such that $X(v_0, v_0g)\beta\pi = H(1 - g)$ for all $g \in G$.

Conversely it is easy to see that (6) implies (5) in general since $G \rightarrow R[E(X)]$, $g \mapsto X(v_0, v_0g)$, is a derivation with kernel H .

Thus it remains to transform (6) into the desired form (3); for convenience let us record what we are proving. (We do not need G finitely generated over H for this step.)

1.1. LEMMA. *Let G act on a tree X with X/G finite, with finite edge stabilizers and with a vertex v_0 having stabilizer precisely H . Then the following are equivalent:*

(6') *There exists a right RG -linear map $\beta: R[E(X)] \rightarrow RG$ such that $X(v_0, v_0g)\beta\pi = H(1 - g)$ for all $g \in G$.*

(3') *$|H \cap H^g| \in R^{-1}$ for all $g \notin H$; and for each vertex $v \neq v_0$ having $\text{star}(v)$ infinite $|H \cap G_v| \in R^{-1}$.*

PROOF. (6') $\Rightarrow$ (3'). Consider any $g \notin H$, and let $x = (X(v_0, v_0g))\beta \in RG$. Then x is stabilized by $G_{v_0} \cap G_{v_0g} = H \cap H^g$ so the coefficient of H in $x\pi \in R[H \setminus G]$ lies in $R[H \cap H^g]$. But the coefficient of H in $x\pi = H(1 - g)$ is 1 so $|H \cap H^g| \in R^{-1}$ as desired.

Now consider any vertex $v \neq v_0$ with $\text{star}(v)$ infinite and let $x = (X(v_0, v))\beta \in RG$. Then x is stabilized by $G_{v_0} \cap G_v = H \cap G_v$ so the coefficient of H in $x\pi \in R[H \setminus G]$ lies in $R[H \cap G_v]$. But for all $g \in G_v$,

$$x\pi(1 - g) = (X(v_0, v)(1 - g))\beta\pi = (X(v_0, v_0g))\beta\pi = H(1 - g)$$

so $x\pi - H$ is fixed by G_v . But G_v is infinite since $\text{star}(v)/G_v \simeq \text{star}(vG) \subseteq X/G$ is finite. Thus $(H \cap G_v) \setminus G_v$ is infinite so the coefficient of H in $x\pi - H$ must be 0; hence the coefficient of H in $x\pi$ is 1 and $|H \cap G_v| \in R^{-1}$ as desired.

(3') $\Rightarrow$ (6'). Let $T = \{x \in X: |H \cap G_x| \notin R^{-1} \text{ or } x = v_0\}$. Notice that T is a subtree of X and that H acts on T stabilizing the vertex v_0 , so T/H is also a tree. By (3'), $\text{star}_X(v)$ is finite for each $v \in V(T) - \{v_0\}$ so T/H is locally finite (in the sense that every vertex has a finite star). We claim that T/H is finite. Suppose it is not. Then by the König Tree Lemma there is an "infinite path" $\{v_0\} = v_0H, (e_1H)^{e_1}, (e_2H_2)^{e_2}, \dots$ in T/H . This lifts back to an "infinite path" $P: v_0, e_1^{e_1}, e_2^{e_2}, \dots$ in T . Now $H \cap G_{e_1} \supseteq H \cap G_{e_2} \dots$ and since these are finite groups the chain is eventually constant. By definition of T we can therefore choose an element h that belongs to every term of this chain and whose order is not in R^{-1} . Since X/G is finite there is some edge e of X such that eG meets P infinitely often. Say $eg_1^{-1}, eg_2^{-1}, \dots$ are distinct edges in P . Then $h \in H \cap G_{eg_i^{-1}}$ for $i = 1, 2, \dots$ so $h^g \in G_e$ for $i = 1, 2, \dots$. But G_e is finite so we must have $h^g = h^g$ for some $i \neq j$. But then $eg_i^{-1}H \neq eg_j^{-1}H$ so $g_i g_j^{-1} \notin H$, while $h^g = h^g \in H^g \cap H^g$. This contradicts $|H \cap H^g| \in R^{-1}$ for all $g \notin H$. Hence T/H is finite as claimed.

Recall that the coboundary of T in X is defined as

$$\delta T = \{e \in E(X) | e \text{ has one vertex in } T \text{ and one vertex not in } T\}.$$

Since $\delta T \subseteq \bigcup_{v \in V(T)} \text{star}(v)$, and $\text{star}(v)$ is finite for all $v \in T - \{v_0\}$, we see $\delta T/H$ is finite. Let E_0 be a (finite) set of representatives in δT for the H -action. For each $e \in E(X)$ set

$$r_e = \begin{cases} |H_e|^{-1} & \text{if } e \in E_0 \text{ and } \iota e \in T, \\ -|H_e|^{-1} & \text{if } e \in E_0 \text{ and } \tau e \in T, \\ 0 & \text{if } e \notin E_0. \end{cases}$$

(Notice that $E_0 \cap T = \emptyset$ so we may view r_e as an element of R .) Define $\beta: R[E(X)] \rightarrow RG$ as the R -linear map with $e\beta = \sum_{g \in G} gr_{eg^{-1}}$. Since almost all the r_e are zero, and edge stabilizers are finite, the support of $e\beta$ is finite; clearly β is RG -linear. The composite $\beta\pi: R[E(X)] \rightarrow R[H \setminus G]$ sends e to $\sum_{Hg \in H/G} Hgr_{eg^{-1}}$ where

$$s_e = \sum_{h \in H} r_{eh^{-1}} = |H_e| \sum_{eh \in eH} r_{eh} = \begin{cases} 1 & \text{if } e \in \delta T \text{ and } \iota e \in T, \\ -1 & \text{if } e \in \delta T \text{ and } \tau e \in T, \\ 0 & \text{if } e \notin \delta T. \end{cases}$$

Thus for any $g_0 \notin H$, if $e_1^{\epsilon_1}, \dots, e_n^{\epsilon_n}$ is the geodesic in X from v_0 to $v_0 g_0$ then $X(v_0, v_0 g_0) \beta \pi = \sum_{Hg \in H/G} Hg(\sum_{i=1}^n \epsilon_i s_{e_i g^{-1}})$. Now $(e_1 g^{-1})^{\epsilon_1}, \dots, (e_n g^{-1})^{\epsilon_n}$ is the geodesic from $v_0 g_0^{-1}$ to $v_0 g_0 g^{-1}$ so

$$\begin{aligned} \sum_{i=1}^n \epsilon_i s_{e_i g^{-1}} &= \begin{cases} +1 & \text{if } v_0 g^{-1} \in T \text{ and } v_0 g_0 g^{-1} \notin T, \\ -1 & \text{if } v_0 g^{-1} \notin T \text{ and } v_0 g_0 g^{-1} \in T, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} +1 & \text{if } g \in H \text{ and } g_0 g^{-1} \notin H, \\ -1 & \text{if } g \notin H \text{ and } g_0 g^{-1} \in H \text{ by (3')}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} +1 & \text{if } Hg = H, \\ -1 & \text{if } Hg = Hg_0 \text{ since } g_0 \notin H, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence $X(v_0, v_0 g_0) \beta \pi = H(1 - g_0)$ as desired.

(The proof shows that (6') and (3') are further equivalent to $|H \cap H^g| \in R^{-1}$ for all $g \notin H$, and $\delta T/H$ is finite.)

In (3), for any vertex v of X consider the action of G_v on $\text{star}(v)$. The kernel of the action is finite since dge stabilizers are finite, and there are only finitely many orbits, since X/G is finite. Thus $\text{star}(v)$ is infinite if and only if G_v is infinite.

Now applying the Bass-Serre Structure Theorem for groups acting on trees ([7, p. 76], cf. [2, p. 21]) we see that (2) and (3) are equivalent.

This completes the proof of Theorem A.

1.2. REMARK. In Theorem A the rôle played by the condition

$$|H \cap H^g| \in R^{-1} \text{ for all } g \notin H$$

is to make $\omega(RH)G$ an RH -summand of $\omega(RG)$. In fact for any group G the following are equivalent:

(a) $\omega(RH)G$ is an RH -summand of $\omega(RG)$.

(b) $|H \cap H^g| \in R^{-1}$ for all $g \notin H$.

(c) $\omega(RG)/\omega(RH)G$ is RH -projective.

PROOF. (a) $\Rightarrow$ (b). Let $\alpha: \omega(RG) \rightarrow \omega(RH)G$ be a splitting. Consider any $g \notin H$ and let $x = (1 - g) - (1 - g)\alpha$. For $h \in H \cap H^g$, say $gh = h'g$, $(1 - g)(1 - h) = (1 - h) - (1 - h')g$ is fixed by α , so $x(1 - h) = 0$. Thus x is fixed by $H \cap H^g$ so the coefficient of H in $x\pi$ lies in $R|H \cap H^g|$. But $x\pi = H(1 - g)$ and the coefficient of H is 1. So $|H \cap H^g| \in R^{-1}$ as claimed.

(b) $\Rightarrow$ (c). There is an isomorphism of RH -modules

$$\omega(RG)/\omega(RH)G \simeq \bigoplus_{HgH \neq H} R[H \setminus H(1 - g)H] \simeq \bigoplus_{HgH \neq H} R[H \cap H^g \setminus H]$$

and the latter is clearly RH -projective if (b) holds.

(c) $\Rightarrow$ (a) is clear.

2. Examples.

2.1. EXAMPLE. Let us illustrate the proof of 1.1 with the simplest case of (2), where H is finite and the graph of groups is just

$$\begin{array}{ccccc} H & & H & & G \\ \xrightarrow{\quad} & & \xrightarrow{\quad} & & \\ v_0 & e & & & v_1 \end{array}$$

Notice that the fundamental group is $H *_H G = G$. The corresponding tree X is

where $H \setminus G = \{H, Hg_2, \dots, Hg_n, \dots\}$. There are two possibilities: either $|H| \in R^{-1}$ or $|H| \notin R^{-1}$.

If $|H| \in R^{-1}$ then, in the notation of 1.1, $T = \{v_0\}$, $\delta T = \{e\}$, $E_0 = \{e\}$ and the map $\beta: R[E(X)] \rightarrow RG$ sends e to $x = |H|^{-1} \sum_{h \in H} h$. For each $g \in G$, $X(v_0, v_0g)\beta = (e - eg)\beta = x(1 - g)$. The splitting $\alpha: \omega(RG) \rightarrow \omega(RH)G$ is given by left multiplication by the idempotent $1 - x \in (RH)G$, and $x\omega(RH) = 0$.

If $|H| \notin R^{-1}$ then we impose the condition $|H \cap H^g| \in R^{-1}$ for all $g \notin H$; this implies $T = \{v_0, e, v_1\}$, $\delta T = \{eg_n | n = 2, 3, \dots\}$ and $E_0 = \{eg_n | i = 2, 3, \dots\}$ where $H \setminus G/H = \{H, Hg_{n_2}H, \dots, Hg_{n_i}H, \dots\}$. The condition that E_0 be finite is the same as $H \setminus G/H$ being finite, i.e. G is finite. The map $\beta: R[E(X)] \rightarrow RG$ sends e to $x = -\sum_i |H \cap H^{g_{n_i}}| \sum_{g \in Hg_{n_i}} g^{-1}$. Then as in the previous case, the splitting $\alpha: \omega(RG) \rightarrow \omega(RH)G$ is given by left multiplication by $1 - x$ where $(1 - x)\omega(RG) \subseteq \omega(RH)G$, and $x\omega(RH) = 0$. (Of course if $H = G$ then $x = 0$.)

Let us note two situations where this simplest case of (2) occurs.

2.2. COROLLARY (COHEN [5, P. 59]). *If G is finite then $\omega(RH)G$ is an RG -summand of $\omega(RG)$ if and only if $|H \cap H^g| \in R^{-1}$ for all $g \notin H$.*

(One direction is given by 1.2, and the other direction by 2.1.)

2.3. COROLLARY. *If G is infinite, finitely generated over H , and finitarily indecomposable over H then $\omega(RH)G$ is an RG -summand of $\omega(RG)$ if and only if either $H = G$ or $|H| \in R^{-1}$.*

PROOF. If $\omega(RH)G$ is an RG -summand of $\omega(RG)$ then there is a finitary decomposition of G over H as in (2), and by the indecomposability of G over H , G must occur as one of the (infinite) vertex groups so either $H = G$ or $|H \cap G| \in R^{-1}$. The converse is clear from 2.1.

2.4. EXAMPLE. In the situation of Example 2.1, the conditions in (2) reduced to relatively easily understood restrictions on H and G . Let us see what can be said in the next simplest case of (2), where the graph of groups is

$$\begin{array}{ccccc} H & & A & & K \\ & \longrightarrow & & & \end{array}$$

with A finite, so the fundamental group G is the coproduct with amalgamation $H *_A K$. Here the conditions (2) state that $|H \cap H^g| \in R^{-1}$, and if K is infinite then $|H \cap K^g| \in R^{-1}$ for all $g \in G$. In particular $A = H \cap K$ and, for $k \in K - A$, $A \cap A^k = (H \cap K) \cap (H \cap K)^k = H \cap H^k$ so we see

(7) if K is infinite then $|A| \in R^{-1}$, and if K is finite then $|A \cap A^k| \in R^{-1}$ for all $k \in K - A$.

Conversely, (7) suffices, since for any $g \notin H$, $H \cap H^g \subseteq A \cap A^k$ for some $k \in K - A$, and for any $g \in G$, $H \cap K^g \subseteq A^h$ for some $h \in H$.

Notice that by 2.1, (7) implies $\omega(RA)K$ is an RK -summand of $\omega(RK)$, so $\omega(RA)G$ is an RG -summand of $\omega(RK)G$. Further, Lewin [6] showed that the natural diagram of RG -modules

$$\begin{array}{ccc} & \omega(RK)G & \\ \omega(RA)G \swarrow & & \searrow \omega(RG) \\ & \omega(RH)G & \end{array}$$

is a pushout and pullback. This makes it clear why $\omega(RH)G$ should be an RG -summand of $\omega(RG)$ in this case.

In the special case where $A = 1$ and $G = H * K$ we have $\omega(RG) = \omega(RH)G \oplus \omega(RK)G$, which is a very natural way for $\omega(RH)G$ to be an RG -summand of $\omega(RG)$. For H torsion-free this is virtually the only way.

2.5. COROLLARY (DUNWOODY [4] AFTER COHEN [1, §7]). *Let G be finitely generated over H , and let H be torsion-free. Then $\omega(RH)G$ is an RG -summand of $\omega(RG)$ if and only if H is a free factor of G .*

PROOF. We have just observed one direction. Conversely, suppose $\omega(RH)G$ is an RG -summand of $\omega(RG)$. Then G can be expressed as the fundamental group of a graph of groups $(\mathcal{G}, Y)$ with Y finite, with finite edge groups, and with $H = \mathcal{G}(v_0)$, one of the vertex groups. (This is immediate from Dunwoody's Theorem, and does not require 1.1.) Suppose that $\mathcal{G}$ has been chosen so that Y is as small as possible. It is not difficult to see that each edge of Y has v_0 as one (or more) of its vertices, so in particular every edge group must be trivial since H has no nontrivial finite subgroups. It follows that H must be a free factor of G , as desired.

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