A CANONICAL FORM FOR PLANAR FAREY SETS

NORMAN RICHERT

ABSTRACT. Asmus Schmidt has developed a new algorithm for generating Gaussian rational approximations to complex irrational numbers. The method is based on the structure of a certain collection of planar sets. Schmidt suggested an inductive proof for the structure theorem without publishing the details. The present paper presents a proof, which is not inductive in character.

In a 1975 *Acta Mathematica* paper [3], Asmus Schmidt developed a new method for generating Gaussian rational approximations to complex irrational numbers. The approximations correspond to decreasing sequences of planar sets, which can be viewed as converging to a number ξ . These sets, which are called *Farey sets*, in turn correspond to complex unimodular transformations. Each Farey set has a canonical representation in terms of a composition of elements of a special set of seven unimodular maps. Schmidt's method can be viewed as analogous to the classical continued fraction algorithm for real numbers, with a sequence of the seven special maps corresponding to the usual sequence of partial quotients. Each complex irrational number has at most 2 sequences of maps associated with it. These sequences (called chains) are periodic if ξ is quadratic and can be used to find the fundamental solution to the Pellian equation $X^2 - DY^2 = \pm 1$, to mention but two analogies to continued fractions.

This paper contains a new proof [2] of the canonical form for Farey sets, on which Schmidt's method depends. The relevant definitions are introduced first, together with four fundamental lemmas.

Let G be the set of unimodular transformations, i.e. complex bilinear maps of the form $m: z \rightarrow (az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{Z}[i]$ and $\det m = ad - bc \in \mathcal{U} = \{\pm 1, \pm i\}$. Let G_p and G_i partition G into maps with $\det m = \pm 1$ and $\det m = \pm i$ respectively. Let the closed upper half-plane be denoted by $\mathfrak{T}$, let $\mathfrak{T}^*$ denote the set

$$\mathfrak{T}^* = \left\{ z: 0 < \operatorname{Re} z < 1, \operatorname{Im} z > 0, N\left(z - \frac{1}{2}\right) > \frac{1}{4} \right\},$$

and let $\overline{\mathfrak{T}}$ and $\overline{\mathfrak{T}^*}$ be the conjugate sets.

For each $m \in G_p$, the *circular Farey set* $F(m)$ is one of the two sets $m(\mathfrak{T})$ or $m(\overline{\mathfrak{T}})$. The choice is made so that $F(m)$ is bounded if $m(\mathbb{R})$ is a circle and is bounded below if $m(\mathbb{R})$ is a line. Recall that G maps circles to circles, where lines are considered to be degenerate circles. It is not hard to show that if $m(\mathbb{R})$ is a line, then it is horizontal, and an integral distance from $\mathbb{R}$.

Received by the editors February 13, 1980. Presented at the January 1981 Annual Meeting in San Francisco.

1980 *Mathematics Subject Classification*. Primary 10F20.

© 1981 American Mathematical Society
0002-9939/81/0000-0458/\$02.00

For each $m \in G_i$, the *triangular Farey set* $F(m)$ is one of the two sets $m(\mathcal{T}^*)$ or $m(\bar{\mathcal{T}}^*)$, where the choice is made so that $F(m)$ is inside the bounded region defined by $m(\mathbf{R})$ if $m(\mathbf{R})$ is a circle, and does not cross the imaginary axis if $m(\mathbf{R})$ is a line (such lines are vertical, at an integral distance from the imaginary axis).

Let $\mathcal{F}$ denote the collection of all circular and triangular Farey sets. It might be noted that all Ford circles and mesh triangles [1] are contained in $\mathcal{F}$. It is a remarkable fact, and crucial to Schmidt's method, that there is a canonical representation for elements of $\mathcal{F}$. This representation is based on the seven unimodular maps in $t_F = \{v_1, v_2, v_3, e_1, e_2, e_3, c\}$, where the corresponding matrices are

$$\begin{aligned} V_1 &= \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, & V_2 &= \begin{pmatrix} 0 & 1 \\ -i & 1 \end{pmatrix}, & V_3 &= \begin{pmatrix} 1-i & i \\ -1 & 1+i \end{pmatrix}, \\ E_1 &= \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}, & E_2 &= \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}, & E_3 &= \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}. \end{aligned}$$

The Farey sets $F(t)$ for $t \in t_F$ are denoted $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and $\mathcal{C}$ respectively. These sets are illustrated in Figure 1. Note, for example, that $\mathcal{V}_1 = F(v_1) = v_1(\mathcal{T})$ and $\mathcal{E}_3 = F(e_3) = e_3(\mathcal{T}^*)$.

FIGURE 1. The seven basic Farey sets

Let T_F denote the matrices corresponding to t_F and $E_F = \{E_1, E_2, E_3\}$. A *regular product* is a product $M_n = T_0 T_1 \cdots T_n$ satisfying

- (i) $T_0 = V_1^{b_0}$, $b_0 \in \mathbb{Z}$, $T_1 \neq V_1$;
- (ii) $T_m \in T_F$ if $\det(T_0 T_1 \cdots T_{m-1}) = \pm 1$;
- (iii) $T_m \in T_F \setminus E_F$ if $\det(T_0 T_1 \cdots T_{m-1}) = \pm i$.

The notation m_n or M_n will refer to a regular product. It can be shown [3] that, for $m_n \in G_p$, $F(m_n) = m_n(\mathcal{T})$ and, for $m_n \in G_i$, $F(m_n) = m_n(\mathcal{T}^*)$.

The Farey sets $F(m_n)$ can be thought of as describing a structure of subdivisions of the complex plane. In particular, if $m_{n-1} \in G_p$, then $F(m_{n-1} \circ t_n)$, for $t_n \in t_F$,

subdivides $F(m_{n-1})$ into 3 circular Farey sets (the $F(m_{n-1} \circ v_j)$) and 4 triangular Farey sets (the $F(m_{n-1} \circ e_j)$ and $F(m_{n-1} \circ c)$). These sets intersect only on their boundaries, the 3 circles are pairwise tangent and tangent to $F(m_{n-1})$ (which is circular). If $m_{n-1} \in G_i$, then $F(m_{n-1} \circ t_n)$, for $t_n \in t_F \setminus e_F$, defines a subdivision of $F(m_{n-1})$ into 1 circular Farey set ($F(m_{n-1} \circ c)$) and 3 triangular Farey sets (the $F(m_{n-1} \circ v_j)$). The set $F(m_{n-1} \circ c)$ is tangent to the 3 sides of $F(m_{n-1})$.

The following two lemmas are found in [3]. The first uses the concept of the *norm* of a Farey set. Each Farey set $F(m)$ has an associated triple of points $m(\infty)$, $m(0)$, and $m(1)$ on its boundary, a *Farey triangle*. If m has the matrix

$$M = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix},$$

then these points are simply p_1/q_1 , p_2/q_2 , and $(p_1 + p_2)/(q_1 + q_2) = p_3/q_3$, respectively. The norm of the Farey triangle is defined by $N(FT) = N(q_1) + N(q_2) + N(q_3)$, where the N on the right-hand side denotes the complex norm. The norm of a Farey set $F(m_n)$ is simply the norm of the associated Farey triangle (the more general $N(F(m))$ is somewhat more complicated).

LEMMA 1. *If m_n is regular, then*

$$\min\{N(F(m_n \circ t_{n+1}))\} > N(F(m_n)),$$

with equality iff $N(F(m_n)) = 2$.

LEMMA 2. *If m_n is regular, then*

$$\text{diam } F(m_n) \leq 4(N(F(m_n)))^{-1/2}$$

if $N > 2$.

It follows from these lemmas that for an infinite sequence of Farey sets of the form $F(t_0) \supseteq F(t_0 \circ t_1) \supseteq F(t_0 \circ t_1 \circ t_2) \supseteq \dots$, if the norm 2 is ever exceeded, then the sequence has exactly one point in its intersection. Hence arises the correspondence between a number ξ and a sequence of regular products.

Let $\mathcal{F}_n = \{F(m_n): m_n \text{ is a regular product}\}$. The $\mathcal{F}_n$ are all disjoint, as is seen from the structure of subdivisions that they produce. The main result is that every Farey set is found in exactly one of the $\mathcal{F}_n$.

The following two lemmas are found in [2].

LEMMA 3. *If $m \in G$ is given by $m: z \rightarrow (p_1 z + p_2)/(q_1 z + q_2)$, then the center and radius of $m(\mathbf{R})$ are given by*

$$c(m) = \frac{p_1 \overline{q_2} - p_2 \overline{q_1}}{q_1 \overline{q_2} - q_2 \overline{q_1}}, \quad r(m) = \frac{1}{i(q_1 \overline{q_2} - q_2 \overline{q_1})}.$$

It follows from Lemma 3 and the fact that the p_i , q_i are integral that $m(\mathbf{R})$ intersects $\mathbf{R}$ in a point of tangency, or intersects in 2 points and $c(m) \in \mathbf{R}$, or else does not intersect $\mathbf{R}$. The following lemma characterizes the two intersection cases.

LEMMA 4. *If $m \in G$ and $m(\mathbf{R})$ is tangent to $\mathbf{R}$, then $m \in G_p$. If $c(m) \in \mathbf{R}$, then $m \in G_i$.*

The main result can now be proved.

THEOREM. Every Farey set is in a unique $\mathcal{F}_n$; i.e. $\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n$, where the union is a disjoint one.

PROOF. Suppose $F = F(m)$ is a circular Farey set and $F \notin \mathcal{F}_n$ for any n . It can be assumed WLOG that $F \subset \mathcal{F}$, since if F has a canonical representation, then $v_1^{b_0}(F)$ clearly does also. There exists an $n \in N_0$, and a finite sequence of Farey sets given by regular products, such that $F(t_0) \supset F(t_0 \circ t_1) \supset F(t_0 \circ t_1 \circ t_2) \supset \cdots \supset F(t_0 \circ t_1 \circ \cdots \circ t_n) \supseteq F$, and $F(t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n \circ t_{n+1}) \not\supset F$ for any $t_{n+1} \in t_F$. This follows from Lemmas 1 and 2 and the nature of the regular product subdivisions. There is no collection of sets $F(m_n \circ t_{n+1})$ (where $m_n = t_0 \circ t_1 \circ t_2 \circ \cdots \circ t_n$) whose union could be F . Hence F must "overlap" one of the $F(m_n \circ t_{n+1})$, in the sense that $(F \cup F(m_n \circ t_{n+1})) \setminus (F \cap F(m_n \circ t_{n+1}))$ is not contained in either $F(m_n \circ t_{n+1})$ or F . In fact it must overlap a circular set $F(m_n \circ t_{n+1})$, since every triangular set $F(m_n \circ t_{n+1})$ shares its concave sides with circular sets. But then $m_{n+1}^{-1}(m_{n+1}(\mathcal{F})) = \mathcal{F}$ and $m_{n+1}^{-1}(m(\mathcal{F}))$ is a circular Farey set overlapping $\mathcal{F}$, which is ruled out by Lemma 4.

Suppose now that F is triangular and assume WLOG that $F = m(\mathcal{F}^*)$. The sides of F are arcs of $m \circ e_j(\mathbb{R})$, $j = 1, 2, 3$, since the sides of $\mathcal{F}^*$ are arcs of $e_j(\mathbb{R})$. But $m \circ e_j \in G_p$, so the Farey sets $F(m \circ e_j)$ are all circular Farey sets, and hence have representations in terms of regular products. Let these three circular Farey sets be $F_1 = (m_{n_1})$, $F_2 = (m_{n_2})$, and $F_3 = (m_{n_3})$, where $n_1 \leq n_2 \leq n_3$.

The F_j determine 2 triangular regions. One is $F(m) = m(\mathcal{F}^*)$ and the other is $m(\bar{\mathcal{F}}^*)$; the latter is not a Farey set, since $m(\bar{\mathcal{F}}^*)$ determines $m(\mathcal{F})$, on which the choice between $m(\mathcal{F}^*)$ and $m(\bar{\mathcal{F}}^*)$ depends in the definition of a triangular Farey set. It follows from the subdivision process induced by regular maps that one of $m(\mathcal{F}^*)$ and $m(\bar{\mathcal{F}}^*)$ is a triangular Farey set associated with a regular product m_n . Hence $F(m) = F(m_n)$, and so F corresponds to a regular product. This completes the proof.

The content of this paper formed part of a doctoral thesis at the Claremont Graduate School. I would like to express appreciation to Professor William J. LeVeque, under whom my program was begun, to Professor Jerome Spanier, who provided a helpful push to complete the work, and to Professor Edward Thurber, who helped to clarify ideas.

REFERENCES

1. Lester R. Ford, *Rational approximations to irrational complex numbers*, Trans. Amer. Math. Soc. **19** (1918), 1–42.
2. Norman Richert, *Diophantine approximation of complex numbers*, unpublished thesis, Claremont Graduate School, 1980.
3. Asmus Schmidt, *Diophantine approximation of complex numbers*, Acta Math. **134** (1975), 1–85.

DEPARTMENT OF MATHEMATICS, LOYOLA MARYMOUNT UNIVERSITY, LOS ANGELES, CALIFORNIA 90045

Current address: Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, Wisconsin 53233