GENERATING DENSE SUBGROUPS OF MEASURE PRESERVING TRANSFORMATIONS

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ABSTRACT. Except for a set of first category, all pairs of measure preserving transformations generate a dense subgroup of G , the group of all invertible measure preserving transformations of the unit interval when G has the weak topology.

Consider G the group of invertible (Lebesgue) measure preserving transformations of the unit interval onto itself. On G , we put the weak topology (Halmos [2, pp. 61–67]). It has been asked by A. Iwanik, whether the group generated by certain pairs of measure preserving transformations is dense in G . The purpose of this note is to prove: *Except for a set of first category in $G \times G$, all pairs of measure preserving transformations generate a subgroup dense in G .*

Related work has been done by R. Grzaślewicz [1] who showed that for any two positive irrational numbers α and β , less than 1, the following transformations S_α and T_β generate a dense subgroup of G , where

$$S_\alpha(x) = x + \alpha \pmod{1}$$

and

$$T_\beta(x) = \begin{cases} x + \beta \pmod{\frac{1}{2}}, & 0 \leq x < \frac{1}{2}, \\ x, & \frac{1}{2} \leq x < 1. \end{cases}$$

It is a classical result of Schreier and Ulam [3] that for a compact connected second countable topological group the pairs of elements generating a dense subgroup is residual $\mathcal{G}_\delta$. Note in our case G is not compact.

PRELIMINARY REMARKS. We shall follow the terminology used by Halmos [2, pp. 61–67]. A dyadic subbasic neighborhood for $S \in G$ is a set of the form $\{T \in G: m(SD \triangle TD) < \varepsilon\}$, where D is a dyadic set (i.e. a finite union of dyadic intervals), m denotes ordinary Lebesgue measure, $\triangle$ denotes symmetric difference, and $\varepsilon > 0$ is a rational number. These sets constitute a subbasis for the weak topology of G , and finite intersections of them constitute a basis. With the weak topology G is a complete metrisable group. By a (cyclic) dyadic permutation of rank n in G , we mean an invertible measure preserving transformation that maps each dyadic

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interval of rank n onto a dyadic interval of rank n by ordinary translation (in one cycle). The fundamental result is the following Weak Approximation Theorem of Halmos [2, p. 65]. *Every dyadic neighborhood contains cyclic permutations of all sufficiently high ranks.* Note that we have written "all sufficiently high ranks" instead of "arbitrarily high rank" as in [2]. This slight change is implicit in the proof in [2]. Let $\{N_i: i = 1, 2, \dots\}$ be any countable base for G (e.g. dyadic neighborhoods centered at dyadic permutations).

PROOF OF THE MAIN RESULT. For each $i = 1, 2, \dots$ set

$$\mathcal{D}_i = \{(S, T) \in G \times G: \text{gp}\{S, T\} \cap N_i \neq \emptyset\}$$

where by $\text{gp}\{S, T\}$ we mean the group generated by S and T . It is clear that each $\mathcal{D}_i$ is open (G is a topological group). The set $\mathcal{D} = \bigcap_{i \geq 1} \mathcal{D}_i$ consists of all pairs (S, T) such that $\text{gp}\{S, T\}$ is dense in G . Since $G \times G$ is a Baire space, we need only show that each $\mathcal{D}_i$ is dense in $G \times G$. Any open set in $G \times G$ contains one of the form $N_j \times N_k$. Thus, to show $\mathcal{D}_i$ is dense we must find (S, T) in $N_j \times N_k$ such that $\text{gp}\{S, T\} \cap N_i \neq \emptyset$.

Choose an integer n sufficiently large so that

- (1) $P_i \in N_i, P_j \in N_j, P_k \in N_k$ are all cyclic dyadic permutations of rank n , and
- (2) if we change P_k on a set of measure not more than $6/2^n$, then the resulting transformation will still be in N_k .

Number the dyadic intervals of rank n in the order that P_j permutes them; thus

$$P_j(D_1) = D_2, P_j(D_2) = D_3, \dots, P_j(D_{2^n-1}) = D_{2^n} \text{ and } P_j(D_{2^n}) = D_1$$

where the D 's are the dyadic intervals of rank n . Set $S = P_j$ and consider $\sigma = (1, 2, \dots, 2^n)$ the corresponding cyclic permutation in $\mathfrak{S}_{2^n}$, the symmetric group on 2^n letters.

P_k also permutes $D_1, D_2, \dots, D_{2^n}$ in one cycle and let $\tau = (t_1, t_2, \dots, t_{2^n})$ be the corresponding cyclic permutation in $\mathfrak{S}_{2^n}$. Consider $\tau' \in \mathfrak{S}_{2^n}$ which differs from τ only in the following ways

- $\tau'(1) = 2, \tau'(2) = 1$ and
- τ' fixes the symbol 2^n .

Thus if we let τ' be the permutation induced by τ elsewhere, then it is easy to verify

$$\{t: \tau(t) \neq \tau'(t)\} \subseteq \{1, 2, \tau^{-1}(1), \tau^{-1}(2), 2^n, \tau^{-1}(2^n)\}.$$

Now, τ' is the product of the transposition $(1, 2)$ and a $2^n - 3$ cycle. Thus $(\tau')^{2^n-3} = (1, 2)$. It is a simple exercise in group theory to show that $\mathfrak{S}_{2^n}$ is generated by the cycle $\sigma = (1, 2, \dots, 2^n)$ and the transposition $(1, 2)$. Thus if we set T to be the dyadic permutation of rank n which corresponds to τ' , then some word in S and T will permute the dyadic intervals of rank n in exactly the same way as $P_i \in N_i$. Since T differs from P_k on at most 6 dyadic intervals of rank n , having total measure not more than $6/2^n$, then by our choice of n , $T \in N_k$. Thus we have shown

THEOREM. *Let G , the group of invertible measure preserving transformations of the unit interval, have the weak topology. Then the pairs of elements in $G \times G$ which generate a dense subgroup of G forms a dense $\mathfrak{G}_\delta$ set.*

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