THE RADII OF STARLIKENESS AND CONVEXITY OF CERTAIN NEVANLINNA ANALYTIC FUNCTIONS

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ABSTRACT. We determine the radius of starlikeness and the radius of convexity of analytic functions having the forms $\int_{-1}^1 d\mu(t)/(z-t)$ and $\int_{-1}^1 zd\mu(t)/(1-tz)$ where $\mu(t)$ is a probability measure.

1. Introduction. Let N_1 and N_2 denote the sets of all functions of the form

$$(1) \quad f(z) \equiv \int_{-1}^1 \frac{d\mu(t)}{z-t}$$

and

$$(2) \quad \phi(z) \equiv \int_{-1}^1 \frac{zd\mu(t)}{1-tz},$$

respectively. Here $\mu(t)$ is a probability measure. We adopt the terminology of [13] and say the functions in N_1 (N_2) are Nevanlinna functions of the first (second) kind defined and analytic in the z -plane cut along $[z|[-1, 1]]$ ($[z|(-\infty, -1] \cup [1, \infty))$). We also say that $f(z)$ and $\phi(z)$ given by (1) and (2), respectively, are *associated* whenever the probability measure $\mu(t)$ is the same in both formulas. Of course, $\phi(1/z) \equiv f(z)$.

Distler [2], Dundučenko [3], Reade [6], Tchakaloff [9], [10] and Thale [11] found that the (maximal) domain of univalence for N_1 is the set $[z| |z| > 1]$, and that for N_2 is the unit disc $\Delta \equiv [z| |z| < 1]$. On the other hand, Todorov showed that if the measure $\mu(t)$ in (1) and (2) is strictly increasing, then each $f \in N_1$ is univalent for $[z| |z| > 1] \setminus [z|z = \pm 1]$ and each $\phi \in N_2$ is univalent for $z \in (\bar{\Delta} \setminus [z|z = \pm 1])$ [12]–[14].

In this note, we find the radii of convexity and starlikeness of both N_1 and N_2 . In addition, we find maximal domains of univalence of the set of certain normalized typically-real functions. This last is a result due to Brannan and Kirwan [1].

The results contained in this note parallel similar ones obtained by Wirths [15] for his set T of all functions g of the form

$$g(z) \equiv \int_0^1 \frac{zd\mu(t)}{1-tz}.$$

2. Results. Our results depend on obtaining the minimum values of certain well-known functionals. Rather than use Golusin's now-standard variational method [4], we use the following striking result due to Ruscheweyh [8].

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THEOREM A. Let $g(z, t)$ be continuous on the set $\Delta x[a, b]$, and let g be analytic in z for each fixed t . Let V denote the set of all functions f of the form

$$f(z) \equiv \int_a^b g(z, t) d\mu(t),$$

where $\mu(t)$ is a probability measure, and let V^2 denote that subset of V defined by those $\mu(t)$ that are step functions with at most two jumps. If L_1 and L_2 are continuous linear functionals defined on the set of all functions that are analytic in Δ (with the topology of uniform convergence on compact subsets of Δ), and if $0 \notin L_2(V^2)$, then for each $f \in V$ there corresponds an $f_0 \in V^2$ such that

$$L_1(f)/L_2(f) = L_1(f_0)/L_2(f_0).$$

For the special case that will lead to our first result, we take $L_1(f) \equiv zf'$ and $L_2(f) \equiv f$, so that Theorem A yields the result

$$(3) \quad \min \left[\operatorname{Re} \frac{z\phi'(z)}{\phi(z)} \mid \phi \in N_2 \right] = \min \left[\operatorname{Re} \frac{z\phi'(z)}{\phi(z)} \mid \phi \in N_2^2 \right]$$

for each fixed z , $z \in \Delta$. Here N_2^2 is the subset of N_2 consisting of all functions ϕ having the form

$$(4) \quad \phi(z) \equiv \frac{\gamma z}{1 - t_1 z} + \frac{(1 - \gamma)z}{1 - t_2 z},$$

where $0 < \gamma \leq 1$, $-1 \leq t_1 \leq t_2 \leq 1$. This last leads to a simple proof of the following result.

THEOREM 1. The radius of starlikeness of N_2 is

$$(5) \quad r_s \equiv \sqrt{3(6 - \sqrt{33})} = 0.8753 \dots$$

This result is sharp. The only extremals are the functions

$$(6) \quad \phi_1(z) \equiv \frac{(6 \pm \sqrt{6 + 2\sqrt{33}})z}{12(1 + z)} + \frac{(6 \mp \sqrt{6 + 2\sqrt{33}})z}{12(1 - z)}$$

with corresponding "critical points"

$$(7) \quad z = \sqrt{3(6 - \sqrt{33})} e^{\pm i\theta_0} \quad \text{and} \quad z = \sqrt{3(6 - \sqrt{33})} e^{\pm i(\pi - \theta_0)}$$

respectively, where

$$(8) \quad \cos \theta_0 = \frac{1}{6} \sqrt{6(\sqrt{33} - 1)}, \quad 0 < \theta_0 < \pi/2.$$

PROOF. As indicated above, we need only consider functions of the form (4) in order to solve the extremal problem (3) for fixed z , $z \in \Delta$. With that in mind, we proceed in a manner reminiscent of Kirwan [5] and Wirths [15]. A straightforward calculation using (4) gives us

$$(9) \quad \frac{z\phi'(z)}{\phi(z)} = \frac{1}{1 - t_1 z} + \frac{1}{1 - t_2 z} - \frac{1}{1 - (\gamma t_2 + (1 - \gamma)t_1)z}.$$

If we set $z = re^{i\theta}$ and

$$G(s, \theta) \equiv \operatorname{Re} \frac{1}{1 - se^{i\theta}},$$

then (9) becomes

$$(10) \quad \operatorname{Re} \frac{z\phi'(z)}{\phi(z)} = G(t_1 r, \theta) - G((\gamma t_2 + (1 - \gamma)t_1)r, \theta) + G(t_2 r, \theta).$$

We now list those properties of $G(s, \theta)$ that we will need to complete our proof. Here θ is fixed, $0 \leq \theta < 2\pi$ and s is variable, $-1 \leq s \leq 1$.

(i) For $\theta = 0$, $G(s, \theta)$ is differentiable for $-1 \leq s \leq 1$ and it is an increasing function there, with $G(-1, 0) = 1/2$, and $G(1, 0) = +\infty$.

(ii) For $\theta = \pi$, $G(s, \theta)$ is differentiable for $-1 \leq s \leq 1$, and is a decreasing function there, with $G(-1, \pi) = +\infty$ and $G(1, \pi) = 1/2$.

(iii) For $\theta \neq 0, \pi$, $G(s, \theta)$ is differentiable for $-1 \leq s \leq 1$ and is an increasing function for $-1 \leq s \leq s_*(\theta)$ for some $s_*(\theta)$, with $G(-1, \theta) = 1/2$ and

$$\max_{-1 \leq s \leq 1} G(s, \theta) \equiv G(s_*(\theta), \theta) = \frac{1 + |\sin \theta|}{2|\sin \theta|};$$

moreover, $G(s, \theta)$ is a decreasing function for $s_*(\theta) \leq s \leq 1$, with $G(1, \theta) = 1/2$. We find

$$(11) \quad s_*(\theta) = \frac{\cos \theta}{1 + |\sin \theta|}.$$

We note that (i) and (ii) can be included in (iii) if we let $s_*(0) = 1$ with $G(s_*(0), 0) = +\infty$, and if we let $s_*(\pi) = -1$ with $G(s_*(\pi), \pi) = +\infty$ in (i) and (ii), respectively.

Recall that we are interested in minimizing the right-hand side of (10), where θ is fixed, r is fixed, $0 < r < 1$, and t_1, t_2 and γ are variable, with $0 \leq \gamma \leq 1$ and $-1 \leq t_1 \leq \gamma t_2 + (1 - \gamma)t_1 \leq t_2 \leq 1$. Hence, using (i), (ii), and (iii) we can make the following assertions concerning the function

$$m(z) \equiv \inf \left[\operatorname{Re} z \frac{\phi'(z)}{\phi(z)} \mid \phi \in N_2^2, z \text{ fixed} \right].$$

(a) If $r \leq s_*(\theta) \leq 1$, then $G(s, \theta)$ is increasing in the interval $-r \leq s \leq r$, so that (10) yields

$$m(z) = \min_{-r \leq s \leq r} G(s, \theta) = G(-r, \theta) = \frac{1 + r \cos \theta}{1 + 2r \cos \theta + r^2} > 0.$$

(b) If $-1 \leq s_*(\theta) \leq -r$, then $G(s, \theta)$ is decreasing in the interval $-r \leq s \leq r$, so that (10) yields

$$m(z) = \min_{-r \leq s \leq r} G(s, \theta) = G(r, \theta) = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2} > 0.$$

(c) If $-r \leq s_*(\theta) \leq r$, then $G(s, \theta)$ is *increasing* in the interval $-r \leq s \leq s_*(\theta)$ and *decreasing* in the interval $s_*(\theta) \leq s \leq r$, then (10) yields

$$\begin{aligned} m(z) &= \min_{-r \leq s \leq s_*(\theta)} G(s, \theta) - \max_{-r \leq s \leq r} G(s, \theta) + \min_{s_*(\theta) \leq s \leq r} G(s, \theta) \\ (12) \quad &= G(-r, \theta) - G(s_*(\theta), \theta) + G(r, \theta) \\ &= \frac{1 + r \cos \theta}{1 + 2r \cos \theta + r^2} - \frac{1 + |\sin \theta|}{2|\sin \theta|} + \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2}, \end{aligned}$$

provided $\sin \theta \neq 0$.

From (a), (b), and (c) above it is evident we must use (12) to determine r so that $m(re^{i\theta}) = 0$. We find

$$(13) \quad r = r(\theta) = \left[\frac{(1 - |\sin \theta|) \cos 2\theta + 2|\sin \theta| \sqrt{|\sin \theta|(|\sin^3 \theta| + 2 \cos^2 \theta)}}{1 + |\sin \theta|} \right]^{1/2}$$

where $\sin 2\theta \neq 0$. The minimum value of $r(\theta)$ in (13) is attained for

$$(14) \quad |\sin \theta| = \frac{1}{6} [\sqrt{33} - 3],$$

and this leads to the radius of starlikeness (5). (See §3 below.)

All solutions of (14) in the interval $0 < \theta < 2\pi$ are given by $\theta = \theta_0, \pi - \theta_0, \pi + \theta_0, 2\pi - \theta_0$, where θ_0 satisfies (8) and (14).

To obtain the extremal functions, we use (12); this implies that the minimum for $m(z)$ is achieved only for

$$(15) \quad (1 - \gamma)t_1 + \gamma t_2 = s_*(\theta_0)/r, \quad t_1 = -1, t_2 = 1,$$

and

$$(16) \quad (1 - \gamma)t_1 + \gamma t_2 = s_*(\pi - \theta_0)/r, \quad t_1 = -1, t_2 = 1,$$

where, by (8), (11), and (14) we have

$$(17) \quad s_*(\theta_0) = \frac{1}{2} \sqrt{2(\sqrt{33} - 5)}, \quad s_*(\pi - \theta) = -\frac{1}{2} \sqrt{2(\sqrt{33} - 5)}.$$

From (5), (15), (16), and (17) we obtain all possible extremal functions, all of which are given by (6) with corresponding "critical points" (7). This completes our proof of Theorem 1.

To obtain the radius of convexity of N_2 we use a result on typically-real functions due to Kirwan [5].

THEOREM 2. *The radius of convexity of N_2 is*

$$r_c = (\sqrt{2} - 1),$$

which is sharp.

PROOF. If we set $z = re^{i\theta}$ in (2), then

$$\frac{\partial}{\partial \theta} \operatorname{Re} \phi(re^{i\theta}) = -r \sin \theta \int_{-1}^1 \frac{(1 - r^2 t^2)}{|1 - tre^{i\theta}|^4} d\mu$$

holds. It follows that for each r , $0 < r < 1$, $\operatorname{Re} \phi(re^{i\theta})$ is decreasing for $0 < \theta < \pi$ and increasing for $\pi < \theta < 2\pi$. Hence, according to a theorem due to Robertson [7], each $\phi \in N_2$ maps the unit disc onto a domain that is convex in the direction of the imaginary axis. Moreover, each $\phi \in N_2$ is real for $-1 < z < 1$.

Now Kirwan has shown that such functions map the disc $[z] |z| < (\sqrt{2} - 1)$ onto a convex domain [5].

The function $\phi_3(z) \equiv z/(1 - z^2)$ shows that our result is sharp. This completes our proof.

Theorems 1 and 2 yield analogous results for the class N_1 .

THEOREM 3. *If $f \in N_1$, then f maps the exterior of the unit disc onto a domain whose complement is a closed region that is convex in the direction of the imaginary axis. Moreover, the sharp radii of starlikeness and convexity of N_1 are*

$$R_s = 1/r_s = \frac{1}{3} \sqrt{\sqrt{33} + 6} = 1.1423 \dots,$$

and

$$R_c = 1/r_c = (\sqrt{2} + 1)$$

respectively. The extremal functions are the obvious modifications of the corresponding extremals for N_2 .

The preceding theorems enable us to sharpen slightly some known results concerning the set TR of all typically-real functions $h(z)$, where

$$(18) \quad h(z) \equiv \int_{-1}^1 \frac{z d\mu(t)}{1 - 2tz + z^2},$$

where $\mu(t)$ is a probability measure.

If we set $w = \frac{1}{2}(z + 1/z)$ in (18), then we obtain

$$h(z) \equiv H(w) \equiv \frac{1}{2} \int_{-1}^1 \frac{d\mu(t)}{w - t}.$$

Our Theorem 3 now yields the following result.

THEOREM 4. *There are two disjoint maximal domains of univalence for the class TR , namely*

$$D_1 \equiv \left[z \mid \left| z + \frac{1}{z} \right| > 2, |z| < 1 \right], \quad D_2 \equiv \left[z \mid \left| z + \frac{1}{z} \right| > 2, |z| > 1 \right].$$

Moreover, each h in TR maps

$$D_3 \equiv \left[z \mid \left| z + \frac{1}{z} \right| > \frac{2}{3} \sqrt{6 + \sqrt{33}}, |z| < 1 \right]$$

and

$$D_4 \equiv \left[z \mid \left| z + \frac{1}{z} \right| > \frac{2}{3} \sqrt{6 + \sqrt{33}}, |z| > 1 \right]$$

onto the complement of a starlike domain, and maps

$$D_5 \equiv \left[z \mid \left| z + \frac{1}{z} \right| > 2(\sqrt{2} + 1), |z| < 1 \right]$$

and

$$D_6 \equiv \left[|z| \left| z + \frac{1}{z} \right| > 2(\sqrt{2} + 1), |z| > 1 \right]$$

onto the complement of a convex domain.

These results include the results that the images $h(D_1)$ and $h(D_2)$ are convex in the direction of the imaginary axis. We note that the domain D_1 is the intersection of the discs centered at $z = \pm i$, radius $\sqrt{2}$, and D_2 is its inverse in the circle $|z| = 1$. D_1 was found by Brannan and Kirwan [1].

3. Additional details. In this section we provide most of the details that lead from (12) and (13) to (14) and (5).

First, $r(\theta) = 1$ holds when $\sin \theta = 0$. Second, $m(re^{i\theta}) = 0$, for r given by (5) and $|\sin \theta|$ given by (14). Hence, the extremum r_s we seek satisfies $0 < r_s < 1$.

For simplicity we set $R \equiv R(x) \equiv [r(\theta)]^2$, with $x \equiv |\sin \theta|$. From (12) and (13), it follows that $R(x)$, such that $m(re^{i\theta}) = 0$, satisfies

$$(19) \quad (1+x)R^2 + 2(-1+x+2x^2-2x^3)R + (1-3x) = 0,$$

and the minimum of $R(x)$, $0 < x < 1$, occurs where $R'(x) = 0$. From (19) we obtain

$$(20) \quad \frac{dR}{dx} = \frac{3 + 2(-1-4x+6x^2)R - R^2}{2[(1+x)R + (-1+x+2x^2-2x^3)]} \equiv \frac{N}{D}.$$

If we use (13), with $R \equiv r^2$, then the denominator D in (20) satisfies the following:

$$\begin{aligned} D &\equiv \frac{1}{R} \left\{ \frac{[(1-x-2x^2-2x^3) + 2x\sqrt{2x-2x^3+2x^4}]^2 + (-1+2x+3x^2)}{(1+x)} \right\} \\ &> \frac{1}{R} \frac{(1-x-2x^2+2x^3)^2 + (-1+2x+3x^2)}{(1+x)} \equiv \frac{4x^3(2-2x^2+x^3)}{(1+x)}. \end{aligned}$$

A simple calculation now shows that $(2-2x^2+x^3)$, and hence D , is positive for $0 < x < 1$. Hence $\min[R(x)|0 < x < 1]$ occurs where $R'(x) = 0$, that is where

$$(21) \quad 3 + 2(-1-4x+6x^2)R - R^2 = 0,$$

where R and x also satisfy (19). From (19) and (21) we obtain

$$(1+2x-2x^2-2x^3)R = 1$$

which then combines with (21) to yield $4x^3(1-x)(2-3x-3x^2) = 0$. This last equation leads to the value $x = |\sin \theta|$ in (14) as the x -value yielding the minimum of $R(x)$, and hence of $r(\theta)$. This completes our presentation of the "details" referred to above.

4. Conclusion. It would be of some interest to find the radius of convexity of the class TR .

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