IRREDUCIBILITY OF CERTAIN DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF $\mathrm{Sp}(n, \mathbf{R})$

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ABSTRACT. An irreducibility theorem is proved for degenerate principal series representations of $\mathrm{Sp}(n, \mathbf{R})$ induced from unitary characters of a certain maximal parabolic subgroup.

In this paper we prove an irreducibility theorem for representations in a degenerate principal series for the *real* symplectic groups $\mathrm{Sp}(n, \mathbf{R})$. These representations T_χ are induced from unitary characters χ of a certain maximal parabolic subgroup $\bar{H}$. Here is the main result:

THEOREM. *The representation T_χ is irreducible for every nontrivial character χ of $\bar{H}$. If χ is the trivial character then T_χ is the direct sum of two irreducible subrepresentations.*

It should be noted that the analogous result over the *complex* field is proved by K. I. Gross in [3], and our methods are analogous. What is surprising in the real case is that, although the index of squares in the real field is two and consequently there are twice as many "orbits" in the real case as in the complex case (cf. below), there is no additional reducibility.

Here is a brief sketch of the steps in the proof. (1) Realize T_χ to act in the space $L^2(Z)$, where Z , the nilpotent radical for the parabolic subgroup H "opposite" to $\bar{H}$, is a 2-step nilpotent subgroup of $\mathrm{Sp}(n, \mathbf{R})$. (2) Apply the Plancherel transform $\mathcal{P}$ of $L^2(Z)$ and consider the unitarily equivalent representation $\hat{T}_\chi = \mathcal{P}T_\chi\mathcal{P}^{-1}$ of $\mathrm{Sp}(n, \mathbf{R})$. It is the transformed representation $\hat{T}_\chi$ that we will actually analyze. (3) Let $\mathcal{Q}(T_\chi)$ denote the von Neumann algebra of the representation and let $\mathcal{Q}'(T_\chi)$ denote the commuting algebra. According to Schur's Lemma, we must determine whether $\mathcal{Q}'(T_\chi)$ consists solely of scalar multiples of the identity operator. Thus, let $B \in \mathcal{Q}'(T_\chi)$, so $\mathcal{P}B\mathcal{P}^{-1} \in \mathcal{Q}'(\hat{T}_\chi)$. (4) When $\hat{T}_\chi$ is restricted to H , we can explicitly determine the commuting algebra $\mathcal{Q}'(\hat{T}_\chi|_H)$ of the restriction of $\hat{T}_\chi$ to H . It is 4-dimensional and is spanned by a certain set of operators $\{I, \mathcal{P}J\mathcal{P}^{-1}, \mathcal{P}I_a\mathcal{P}^{-1}, \mathcal{P}J_a\mathcal{P}^{-1}\}$. Here I (the identity), J , I_a , and J_a are operators back on the original representation space $L^2(Z)$; and of course, they form a basis for the commuting algebra $\mathcal{Q}'(T|_H)$. (5) Since H is a *maximal* subgroup of $\mathrm{Sp}(n, \mathbf{R})$, it is enough to analyze just one element of the group outside of H . For

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this purpose, a "Weyl reflection" p is a propitious choice. Thus, for $B = \alpha I + \beta J + \gamma I_a + \delta J_a$, we know that B is in $\mathcal{Q}'(T_\chi)$ if and only if $BT_\chi(p) = T_\chi(p)B$. Finally, one shows, using a Mellin-type analysis on $L^2(Z)$, that this condition is satisfied for nontrivial χ if and only if $\beta = \gamma = \delta = 0$; the condition is satisfied for trivial χ if and only if $\gamma = \delta = 0$.

With reference to [3], what is novel in the real case is the appearance of the convolution operators I_a and J_a on the nilpotent group, and a more intricate Mellin analysis in the last step. In what follows we refer the reader to [3], or [2], for details of proofs that are the same over the real field as for the complex field. We include the details where the real field introduces significant variations, for example in the Mellin analysis.

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1. Reduction with respect to the opposite parabolic. Let $n > 1$ and $m = n - 1$. Represent the real symplectic group $\mathrm{Sp}(n, \mathbf{R})$ as

$$\Sigma_n = \left\{ g \in \mathbf{R}^{2n \times 2n} : gp g' = p, p = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \right\},$$

where g' denotes the transpose of the matrix g . Define the following subgroups of Σ_n :

$$\begin{aligned} A &= \left\{ \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \alpha \in \mathbf{R}, \alpha \neq 0 \right\}, \\ S &= \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s_{11} & 0 & s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & s_{21} & 0 & s_{22} \end{bmatrix} : s = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \in \Sigma_m \right\}, \\ Z &= \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ -y' & I & 0 & 0 \\ t & x & 1 & y \\ x' & 0 & 0 & I \end{bmatrix} : t \in \mathbf{R}, x, y \in \mathbf{R}^{1 \times m} \right\}, \\ \bar{H} &= Z' S A. \end{aligned}$$

It is useful to identify A with $\mathbf{R}^*$ in the obvious way and use α to denote both an element of $\mathbf{R}^*$ and the corresponding element of A . Similarly, we use s for an element of Σ_m and the corresponding element of S . Elements of Z are denoted z or (x, y, t) .

The degenerate principal series representation T_χ induced by a unitary character χ of $\bar{H}$ is given (see [2, (1.1)]) by

$$(1.1) \quad T_\chi(g)f(z) = \delta_{\bar{H}}(zg)^{-1/2} \chi(zg) f(z\bar{g})$$

for $g \in \Sigma_n$ and $f \in L^2(Z)$, where $\delta_{\bar{H}}$ is the modular function of $\bar{H}$. By an easy computation, $\delta_{\bar{H}}(h) = |\alpha|^{-2m-2}$ for $h = z's\alpha \in \bar{H}$, and every unitary character has

the form $\chi(h) = |\alpha|^{\sigma}$ for some $\sigma \in \mathbf{R}$. If $g = z\alpha$ is an element of the opposite parabolic subgroup $H = ZSA$ and $\zeta \in Z$ then the action of Σ_n on Z occurring in (1.1) is given by $\zeta\bar{g} = \zeta(zs\alpha) = \alpha^{-1}s^{-1}\zeta zs\alpha$. In this case, (1.1) becomes

$$(1.2) \quad T_x(zs\alpha)f(\zeta) = |\alpha|^{m+1+i\sigma}f(\alpha^{-1}s^{-1}\zeta zs\alpha), \quad \text{for } f \in L^2(Z).$$

The Plancherel transform of $L^2(Z)$ involves the dual space Λ of Z . Let $V = \mathbf{R}^{1 \times m}$ and $(y|x) = y'x$ for any $x, y \in V$. Identify Λ with $\mathbf{R}^*$ as follows: to $\lambda \in \mathbf{R}^*$, associate the irreducible unitary representation $\hat{\lambda}: Z \rightarrow \mathcal{U}(L^2(V))$ given by

$$(1.3) \quad \hat{\lambda}(z)\phi(u) = \exp[i\lambda\{t - (y|2u + x)\}]\phi(u + x),$$

where $z = (x, y, t) \in Z$ and $\phi \in L^2(V)$. We regard Λ as the set $\{\hat{\lambda}: \lambda \in \mathbf{R}^*\}$ supplied with the Lebesgue measure of $\mathbf{R}^*$.

Let $\mathrm{HS}(L^2(V))$ denote the Hilbert space of Hilbert-Schmidt operators on $L^2(V)$. The Plancherel transform $\mathcal{P}: L^2(Z) \rightarrow L^2(\Lambda, \mathrm{HS}(L^2(V)), d\mu(\lambda))$ is defined for f in a dense subspace of $L^2(Z)$ by $\mathcal{P}f(\lambda) = \int_Z f(z)\hat{\lambda}(z) dz$. The Plancherel measure on Λ , $d\mu(\lambda) = (2\pi)^{-1}\pi^{-m}|\lambda|^m d\lambda$, makes $\mathcal{P}$ an isometry.

The following analogue of Theorem 2 of [3] introduces representations of S and A which play a role in the reduction of $T|_H$:

THEOREM 1.4. *For $\hat{\lambda} \in \Lambda$, $s \in S$, and $\alpha \in A$ define the representation $s\alpha\hat{\lambda}$ of Z by $(s\alpha\hat{\lambda})(z) = \hat{\lambda}(\alpha^{-1}s^{-1}zs\alpha)$. Fix $\lambda \in \mathbf{R}^*$; then (i) $s\alpha\hat{\lambda}$ is unitarily equivalent to $(\lambda\alpha^2)^{\sim}$. (ii) There exists a unique multiplier representation $\tilde{\lambda}$ of S which satisfies $\tilde{\lambda}(s^{-1}zs) = \tilde{\lambda}(s)^{-1}\hat{\lambda}(z)\tilde{\lambda}(s)$ for all $z \in Z$, $s \in S$. (iii) Let D be the representation of A in $L^2(V)$ given by $D(\alpha)\phi(u) = |\alpha|^{m/2}\phi(\alpha u)$. Then D is the unique representation of A , up to tensor product with characters, such that $\hat{\lambda}(\alpha^{-1}z\alpha) = D(\alpha)^{-1}(\lambda\alpha^2)^{\sim}(z)D(\alpha)$ for all $z \in Z$. Also, $(\lambda\alpha^2)^{\sim}(s) = D(\alpha)\tilde{\lambda}(s)D(\alpha)^{-1}$ for all $s \in S$ so that $\tilde{\lambda}$ and $(\lambda\alpha^2)^{\sim}$ are unitarily equivalent. (iv) The operators $D(\pm 1)$ form a basis for the commuting algebra $\mathcal{Q}'(\tilde{\lambda})$.*

We remark that the representation $\tilde{\lambda}$ of $S \cong \mathrm{Sp}(n-1, \mathbf{R})$ is a variant of the oscillator, or metaplectic representation.

Fix a unitary character χ of H and let $T = T_\chi$. Define $\hat{T} = \mathcal{P}T\mathcal{P}^{-1}$; then $\hat{T}$ is a representation of Σ_n unitarily equivalent to T and acting in $\mathcal{H} = L^2(\Lambda, \mathrm{HS}(L^2(V)), d\mu(\lambda))$. For f in a certain dense subspace of $L^2(Z)$ and $\hat{f} = \mathcal{P}f$, $\hat{T}(zs\alpha)\hat{f}$ is given by

$$(1.5) \quad \hat{T}(zs\alpha)\hat{f}(\lambda) = |\alpha|^{-m-1+i\sigma}\tilde{\lambda}(s)D(\alpha)\hat{f}(\alpha^{-2}\lambda)D(\alpha)^{-1}\tilde{\lambda}(s)^{-1}\hat{\lambda}(z)^{-1}$$

for any $zs\alpha \in H$.

Analysis of this formula as in [2, Theorem 3.3] and [3, Theorem 6] yields

THEOREM 1.6. *The commuting algebra $\mathcal{Q}'(\hat{T}|_H)$ is the algebra of all operators on $\mathcal{H}$ of the form $B = \int_\Lambda B(\lambda) \otimes I d\mu(\lambda)$, where $\lambda \rightarrow B(\lambda)$ is a continuous mapping of Λ into $\mathcal{L}(L^2(V))$ satisfying (i) $B(\lambda) \in \mathcal{Q}'(\tilde{\lambda})$ for all $\lambda \in \Lambda = \mathbf{R}^*$, and (ii) $B(\pm\alpha) = D(\alpha)B(\pm 1)D(\alpha)^{-1}$ for all $\alpha \in A = \mathbf{R}^*$.*

Let $\Lambda^+ = (0, \infty)$ and $\Lambda^- = (-\infty, 0)$ be the subsets of Λ determined by the identification with $\mathbf{R}^*$. The two preceding theorems combine to give the basis

$\{I, J, I_a, J_a\}$ for the 4-dimensional algebra $\mathcal{Q}'(\hat{T}|_H)$, where

$$\begin{aligned} \hat{I} &= \text{the identity operator on } \mathcal{K}, \\ \hat{J} &= \int_{\Lambda} D(-1) \otimes I \, d\mu(\lambda), \\ (1.7) \quad \hat{I}_a &= \int_{\Lambda^+} I \otimes I \, d\mu(\lambda) - \int_{\Lambda^-} I \otimes I \, d\mu(\lambda), \quad \text{and} \\ \hat{J}_a &= \int_{\Lambda^+} D(-1) \otimes I \, d\mu(\lambda) - \int_{\Lambda^-} D(-1) \otimes I \, d\mu(\lambda). \end{aligned}$$

The corresponding basis $\{I, J, I_a, J_a\}$ can be found by applying the inverse transform $B = \mathcal{P}^{-1} \hat{B} \mathcal{P}$. Define the following unitary operators on $L^2(Z)$:

$$\begin{aligned} Kf(x, y, t) &= |t|^m f(ty, -tx, t), \\ \mathcal{F}_1 f(x, y, t) &= (2\pi)^{-m/2} \int_V f(u, y, t) \exp i(x|u) \, du, \\ (1.8) \quad \mathcal{F}_2 f(x, y, t) &= (2\pi)^{-m/2} \int_V f(x, u, t) \exp i(y|u) \, du, \\ \mathcal{F}_3 f(x, y, t) &= (2\pi)^{-1/2} \int_{\mathbf{R}} f(x, y, \lambda) \exp i t \lambda \, d\lambda, \\ M_3 f(x, y, t) &= (\operatorname{sgn} t) f(x, y, t), \end{aligned}$$

for f in a suitable dense subset of $L^2(Z)$. Note that $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are the partial (Euclidean) Fourier transforms. In analogy to [3, p. 415] one can prove the following:

THEOREM 1.9. *The commuting algebra $\mathcal{Q}'(T|_H)$ is 4-dimensional with basis $\{I, J, I_a, J_a\}$, where $J = \mathcal{F}_3^{-1} K \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3$, $I_a = \mathcal{F}_3^{-1} M_3 \mathcal{F}_3$, and $J_a = \mathcal{F}_3^{-1} M_3 K \mathcal{F}_1 \mathcal{F}_2 \mathcal{F}_3$.*

In reference to our introductory remarks for the complex group $\operatorname{Sp}(n, \mathbf{C})$, the commuting algebra $\mathcal{Q}'(T|_H)$ in the complex case is just 2-dimensional. The reason for this difference involves the action of A on Λ given by $\lambda \rightarrow \alpha^2 \lambda$. Condition (ii) of Theorem 1.6 states that under this action, $B(\lambda)$ is uniquely determined on each orbit in Λ by evaluation at a fixed representative λ_0 in the orbit. In the real case there are two orbits allowing for two independent choices, say $B(1)$ and $B(-1)$, from $\mathcal{Q}'(\tilde{\Lambda})$. But in the complex case, the action is transitive so that B is uniquely determined by the choice of $B(1)$ from the 2-dimensional algebra $\mathcal{Q}'(\tilde{\Lambda})$. This explains our earlier comment regarding the index of squares in the field.

Although the description of $\mathcal{Q}'(T|_H)$ given in Theorem 1.9 is sufficient for the needs of the next section, it is interesting to view the operators J , I_a , and J_a as convolutions on the nilpotent group Z by generalized functions. Let $\mathcal{S}$ be the space of test functions consisting of all infinitely differentiable, rapidly decreasing functions on Z . Let $\mathcal{S}'$ be the space of continuous linear functionals on $\mathcal{S}$; i.e., the generalized functions or tempered distributions on Z . As is customary, for a locally integrable function ϕ on Z , the formula $(\phi, f) = \int_Z \overline{\phi(z)} f(z) \, dz$ ($f \in \mathcal{S}$) defines an element of $\mathcal{S}'$ which is also called ϕ .

Let $A \in \mathcal{L}(L^2(Z))$ and suppose $A: \mathcal{S} \rightarrow C(Z)$. Define, for each $f \in \mathcal{S}$, $(\phi_A, f) = \overline{Af^*(e)}$, where $f^*(z) = f(z^{-1})$ and e is the identity of Z ; then ϕ_A is a generalized function. If we suppose also that $A\tau(z) = \tau(z)A$ for all right translations $\tau(z)$ of $L^2(Z)$ then A is the generalized convolution given by $Af(z) = \phi_A * f(z) = \overline{(\phi_A, (\tau(z)f)^*)}$, for $f \in \mathcal{S}$. By the usual abuse of notation we can write this as $Af(z) = \int_Z \phi_A(\xi) f(\xi^{-1}z) d\xi$.

The operators J , I_a , and J_a are generalized convolutions. To get some feeling for the generalized functions ϕ_J , ϕ_{I_a} , and ϕ_{J_a} , consider first the operators $A_1 = \mathcal{F}_3^{-1}K\mathcal{F}_1\mathcal{F}_2 = J\mathcal{F}_3^{-1}$, $A_2 = \mathcal{F}_3^{-1}M_3 = I_a\mathcal{F}_3^{-1}$, and $A_3 = \mathcal{F}_3^{-1}M_3K\mathcal{F}_1\mathcal{F}_2 = J_a\mathcal{F}_3^{-1}$. By direct computations,

$$\begin{aligned} \phi_{A_1}(x, y, t) &= (2\pi)^{-m-1/2}|t|^m, \\ (1.10) \quad \phi_{A_2}(x, y, t) &= (2\pi)^{-1/2}\delta(x, y)(\text{sgn } t), \quad \text{and} \\ \phi_{A_3}(x, y, t) &= (2\pi)^{-m-1/2}(\text{sgn } t)|t|^m, \end{aligned}$$

where the formula for ϕ_{A_2} has the meaning $(\phi_{A_2}, f) = \int_{\mathbf{R}} (2\pi)^{-1/2}(\text{sgn } t)f(0, 0, t) dt$, for any $f \in \mathcal{S}$.

If $\phi \in \mathcal{S}'$, define $\mathcal{F}_3^{-1}\phi$ to be the generalized function given by $(\mathcal{F}_3^{-1}\phi, f) = (\phi, \mathcal{F}_3 f)$ ($f \in \mathcal{S}$). Since $\mathcal{F}_3: \mathcal{S} \rightarrow \mathcal{S}$, the definition makes sense.

THEOREM 1.11. *The operators J , I_a , and J_a are convolutions by the generalized functions $\mathcal{F}_3^{-1}\phi_{A_1}$, $\mathcal{F}_3^{-1}\phi_{A_2}$, and $\mathcal{F}_3^{-1}\phi_{A_3}$, respectively where ϕ_{A_1} , ϕ_{A_2} , and ϕ_{A_3} are given by (1.10).*

PROOF. $(\phi_J, f) = \overline{(Jf)^*(e)} = \overline{(A_1\mathcal{F}_3 f)^*(e)} = (\phi_{A_1}, \mathcal{F}_3 f) = (\mathcal{F}_3^{-1}\phi_{A_1}, f)$. Similarly for I_a and J_a .

2. Mellin analysis. It is known that $H = ZSA$ is a maximal subgroup of Σ_n and, therefore, Σ_n is generated by $H \cup \{p\}$, where $p = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. It follows that $B \in \mathcal{Q}'(T)$ if and only if $B \in \mathcal{Q}'(T|_H)$ and B commutes with $T(p)$. By Theorem 1.9 the elements of $\mathcal{Q}'(T|_H)$ have the form

$$(2.1) \quad B = \alpha I + \beta J + \gamma I_a + \delta J_a \quad (\alpha, \beta, \gamma, \delta \in \mathbf{C}).$$

In this section we will show that an operator of the form (2.1) commutes with $T(p)$ if and only if one of the two following cases occurs: (i) χ is a nontrivial character and $\beta = \gamma = \delta = 0$ or (ii) χ is the trivial character and $\gamma = \delta = 0$.

Let the unitary character χ of H be given by $\chi(h) = |\alpha|^{i\sigma}$. From (1.1) the operator $T(p)$ is given by

$$(2.2) \quad T(p)f(x, y, t) = |t|^{m+1-i\sigma}f(-t^{-1}y, t^{-1}x, -t^{-1}),$$

for $f \in L^2(Z)$. Define unitary operators T_3 and $B_3(\chi)$ on $L^2(Z)$ by $T_3 f(x, y, t) = |t|^{-1}f(x, y, -t^{-1})$, $B_3(\chi)f(x, y, t) = |t|^{i\sigma}f(x, y, t)$. Then it is easy to verify that

$$(2.4) \quad T(p) = B_3(\chi)T_3K,$$

where K is defined by (1.8).

All of the relevant operators $T(p)$, J , I_a , and J_a are now expressed in terms of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, M_3 , K , T_3 , and $B_3(\chi)$. The notations for these latter operations have been

chosen, as in [3], so that operators with different subscripts commute. In order to consider a Mellin-type transform of these operators, we need to restrict them to certain subspaces of $L^2(Z)$ which transform under characters of a finite group; namely, the group $G = \bigoplus^{2m+1} \mathbf{Z}_2$, where $\mathbf{Z}_2 = \{-1, 1\}$ under multiplication. View the elements of G as triples (e_1, e_2, e_3) , where e_1 and e_2 are $m \times m$ diagonal matrices with ± 1 in each diagonal position and $e_3 = \pm 1$. Let $\hat{G}$ be the group of characters of G , let $\eta \in \hat{G}$, and define $L^2(Z)^\eta$ to be the subspace of $L^2(Z)$ consisting of all f such that $f(xe_1, ye_2, te_3) = \eta(e_1, e_2, e_3)f(x, y, t)$ for all $(e_1, e_2, e_3) \in G$. It is not difficult to see that $L^2(Z) = \sum_{\eta \in \hat{G}} \bigoplus L^2(Z)^\eta$.

Each of the subspaces $L^2(Z)^\eta$ is invariant under the operations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, T_3$ and $B_3(\chi)$. However, the operator M_3 permutes the subspaces sending $L^2(Z)^\eta$ to $L^2(Z)^{\epsilon\eta}$, where $\epsilon(e_1, e_2, e_3) = e_3$. The operator K also permutes the subspaces. It sends $L^2(Z)^\eta$ to $L^2(Z)^\mu$, where $\mu(e_1, e_2, e_3) = \eta(e_3e_2, e_3e_1, e_3)$. We shall call an element of $\mathcal{L}(L^2(Z))$ a *permutation operator* if it permutes the set $\{L^2(Z)^\eta: \eta \in \hat{G}\}$. Thus, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, T_3, B_3(\chi), M_3$, and K are all permutation operators. Observe that: (a) the permutations M_3 and K have order 2 and (b) the permutations M_3 and KM_3 ($= M_3K$) have no fixed points in the set $\{L^2(Z)^\eta: \eta \in \hat{G}\}$.

Suppose that $\alpha I + \beta J + \gamma I_a + \delta J_a \in \mathcal{Q}'(T)$, with β, γ, δ not all zero, and let $f \in L^2(Z)^\eta$. Then $\beta J + \gamma I_a + \delta J_a \in \mathcal{Q}'(T)$ and, in particular,

$$(2.5) \quad \beta J T(p)f + \gamma I_a T(p)f + \delta J_a T(p)f = \beta T(p)Jf + \gamma T(p)I_a f + \delta T(p)J_a f.$$

According to the preceding discussion, among the 6 terms in (2.5), only $\beta J T(p)f$ and $\beta T(p)Jf$ are in $L^2(Z)^\eta$; the other terms are in other subspaces of $L^2(Z)$. By a direct sum argument, then $\beta J T(p)f = \beta T(p)Jf$. If $\beta \neq 0$ then it follows that $J \in \mathcal{Q}'(T)$. Suppose $\beta = 0$ and $J \notin \mathcal{Q}'(T)$; then $\gamma I_a + \delta J_a \in \mathcal{Q}'(T)$ and hence $(\gamma I_a + \delta J_a)^2 = (\gamma^2 + \delta^2)I + 2\gamma\delta J \in \mathcal{Q}'(T)$, which implies $2\gamma\delta = 0$. Thus, either $I_a \in \mathcal{Q}'(T)$ or $J_a \in \mathcal{Q}'(T)$. The result is that the existence of a nonscalar element of $\mathcal{Q}'(T)$ implies that J, I_a , or J_a is in $\mathcal{Q}'(T)$. Therefore, it suffices to determine separately whether J, I_a or J_a commutes with $T(p)$.

In order to deal with these commutativity questions, we introduce a Mellin-type transform $\mathfrak{M}$ on $L^2(Z)$. For $x = [x_1, x_2, \dots, x_m]$ and $u = [u_1, u_2, \dots, u_m]$ in V , let $u^{-1/2}$ and u^{ix} be defined by

$$u^{-1/2} = u_1^{-1/2} \cdot u_2^{-1/2} \cdot \dots \cdot u_m^{-1/2} \quad \text{and} \quad u^{ix} = u_1^{ix_1} \cdot u_2^{ix_2} \cdot \dots \cdot u_m^{ix_m}.$$

For f in a certain dense subspace of $L^2(Z)$, we define

$$(2.6) \quad \mathfrak{M}f(x, y, t) = c \int_0^\infty \dots \int_0^\infty u^{-1/2} v^{-1/2} \lambda^{-1/2} f(u, v, \lambda) u^{ix} v^{iy} \lambda^{it} du dv d\lambda,$$

where $c = (2\pi)^{-(2m+1)/2}$. From the ordinary Mellin transform $\mathfrak{M}$ on $L^2(\mathbf{R}^+)$, it can easily be shown that (2.6) defines a surjective, bounded linear operator on $L^2(Z)$. Denote by $\mathfrak{M}_\eta$ the restriction of $\mathfrak{M}$ to the subspace $L^2(Z)^\eta$. Then $\mathfrak{M}_\eta: L^2(Z)^\eta \rightarrow L^2(Z)$ is a bijection, so we have a bounded linear transformation $\mathfrak{M}_\eta^{-1}: L^2(Z) \rightarrow L^2(Z)^\eta$ satisfying $\mathfrak{M} \mathfrak{M}_\eta^{-1} = I$, and $(\mathfrak{M}_\eta^{-1} \mathfrak{M})_\eta = \mathfrak{M}_\eta^{-1} \mathfrak{M}_\eta = I_\eta$, where I_η is the identity on $L^2(Z)^\eta$.

Suppose that X is a permutation operator. For fixed $\eta \in \hat{G}$, we define the η -transform of X , denoted X^η , by

$$(2.7) \quad X^\eta = \mathfrak{M}_X \mathfrak{M}_\eta^{-1}.$$

Then $X^\eta \in \mathcal{L}(L^2(Z))$. The value of this transform in our context is suggested by the following two lemmas, the proofs of which are straightforward.

LEMMA 2.8. *Let X and Y be permutation operators and suppose Y sends $L^2(Z)^\eta$ to $L^2(Z)^\nu$. Then*

- (i) $(XY)^\eta = X^\nu Y^\eta$, and
- (ii) $(Y^{-1})^\nu = (Y^\eta)^{-1}$.

LEMMA 2.9. *Let X and Y be permutation operators. Then $XY = YX$ if and only if the associated permutations commute and $(XY)^\eta = (YX)^\eta$ for every $\eta \in \hat{G}$.*

To determine whether J , I_a , or J_a commutes with $T(p)$ we will apply the preceding lemmas. According to Theorem 1.9 and (2.4), the conditions (a) $I_a T(p) = T(p) I_a$, (b) $J_a T(p) = T(p) J_a$, and (c) $J T(p) = T(p) J$ are equivalent to

$$(2.10) \quad \begin{aligned} (a) \quad & \mathfrak{F}_3^{-1} M_3 \mathfrak{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K \mathfrak{F}_3^{-1} M_3 \mathfrak{F}_3, \\ (b) \quad & \mathfrak{F}_3^{-1} M_3 K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K \mathfrak{F}_3^{-1} M_3 K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3, \\ (c) \quad & \mathfrak{F}_3^{-1} K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K \mathfrak{F}_3^{-1} K \mathfrak{F}_1 \mathfrak{F}_2 \mathfrak{F}_3. \end{aligned}$$

By manipulation of these equations and the fact that operators with different subscripts commute, it is verified that (2.10) is equivalent to

$$(2.11) \quad \begin{aligned} (a) \quad & \mathfrak{F}_3^{-1} M_3 \mathfrak{F}_3 B_3(\chi) T_3 K = B_3(\chi) T_3 K \mathfrak{F}_3^{-1} M_3 \mathfrak{F}_3, \\ (b) \quad & B_3(\chi)^{-1} \mathfrak{F}_3^{-1} M_3 K^{-1} \mathfrak{F}_3 B_3(\chi) T_3 = T_3 K^{-1} (\mathfrak{F}_1^{-1})^2 (\mathfrak{F}_2^{-1})^2 \mathfrak{F}_3^{-1} M_3 K^{-1} \mathfrak{F}_3 K^{-1}, \\ (c) \quad & B_3(\chi)^{-1} \mathfrak{F}_3^{-1} K^{-1} \mathfrak{F}_3 B_3(\chi) T_3 = T_3 K^{-1} (\mathfrak{F}_1^{-1})^2 (\mathfrak{F}_2^{-1})^2 \mathfrak{F}_3^{-1} K^{-1} \mathfrak{F}_3 K^{-1}. \end{aligned}$$

This set of conditions is an improvement in that $(\mathfrak{F}_1^{-1})^2 f(x, y, t) = f(-x, y, t)$ and $(\mathfrak{F}_2^{-1})^2 f(x, y, t) = f(x, -y, t)$ are particularly simple operators.

By taking the η -transform of each side of the equations in (2.11), we can use Lemma 2.9 to determine whether these equations hold. First, we need the η -transforms of all the operators which occur in (2.11). For $x \in V$ let $|x| = x_1 + x_2 + \cdots + x_m$.

THEOREM 2.12. *Fix $\eta \in \hat{G}$. The following are η -transforms:*

- (a) $((\mathfrak{F}_1^{-1})^2)^\eta f(x, y, t) = \eta(-I, I, 1) f(x, y, t)$,
- (b) $((\mathfrak{F}_2^{-1})^2)^\eta f(x, y, t) = \eta(I, -I, 1) f(x, y, t)$,
- (c) $K^\eta f(x, y, t) = \eta(I, -I, 1) f(y, x, t - |x| - |y|)$,
- (d) $(K^{-1})^\eta f(x, y, t) = \eta(-I, I, 1) f(y, x, t + |x| + |y|)$,
- (e) $B_3(\chi)^\eta f(x, y, t) = f(x, y, t - \sigma)$,
- (f) $T_3^\eta f(x, y, t) = \eta(I, I, -1) f(x, y, -t)$,
- (g) $M_3^\eta f(x, y, t) = f(x, y, t)$,
- (h) $\mathfrak{F}_3^\eta f(x, y, t) = w_\eta(t) f(x, y, -t)$, and

(i) $(\mathcal{F}_3^{-1})^\eta f(x, y, t) = \eta(I, I, -1)w_\eta(t)f(x, y, -t)$ where

$$w_\eta(t) = \begin{cases} 2^{it}\Gamma\left(\frac{it}{2} + \frac{1}{4}\right)/\Gamma\left(-\frac{it}{2} + \frac{1}{4}\right) & \text{if } \eta(I, I, -1) = 1, \\ i2^{it}\Gamma\left(\frac{it}{2} + \frac{3}{4}\right)/\Gamma\left(-\frac{it}{2} + \frac{3}{4}\right) & \text{if } \eta(I, I, -1) = -1. \end{cases}$$

PROOF. The computations are straightforward except in the case of (h). But one can show that

$$\mathcal{F}_3^\eta f(x, y, t) = \begin{cases} f(x, y, -t)(2\pi)^{-1/2} \int_0^\infty 2(\cos \lambda)\lambda^{it-1/2} d\lambda & \text{if } \eta(I, I, -1) = 1, \\ f(x, y, -t)(2\pi)^{-1/2} \int_0^\infty 2i(\sin \lambda)\lambda^{it-1/2} d\lambda & \text{if } \eta(I, I, -1) = -1. \end{cases}$$

These integrals are the ordinary Mellin transforms of the function $2 \cos t$ and $2i \sin t$ evaluated at $it + 1/2$, which are found in [4, p. 196] and, after simplification, yield formula (h).

Fix $\eta \in \hat{G}$ and let $\varepsilon, \mu \in \hat{G}$ be defined as before by $\varepsilon(e_1, e_2, e_3) = e_3$ and $\mu(e_1, e_2, e_3) = \eta(e_3 e_2, e_3 e_1, e_3)$. From Theorem 2.12 and Lemma 2.8, one obtains the following η -transforms of the expressions in (2.11): From (2.11)(a),

$$(2.13) \quad \begin{aligned} & (\mathcal{F}_3^{-1} M_3 \mathcal{F}_3 B_3(\chi) T_3 K)^\eta f(x, y, t) \\ & = -\eta(I, -I, 1)w_{\varepsilon\mu}(t)w_\mu(-t)f(y, x, \sigma - t - |x| - |y|) \end{aligned}$$

and

$$(2.14) \quad \begin{aligned} & (B_3(\chi) T_3 K \mathcal{F}_3^{-1} M_3 \mathcal{F}_3)^\eta f(x, y, t) = \eta(-I, I, 1)w_\eta(\sigma - t - |x| - |y|) \\ & \cdot w_\eta(t + |x| + |y| - \sigma)f(y, x, \sigma - t - |x| - |y|). \end{aligned}$$

From (2.11)(b),

$$(2.15) \quad \begin{aligned} & (B_3(\chi)^{-1} \mathcal{F}_3^{-1} M_3 K^{-1} \mathcal{F}_3 B_3(\chi) T_3)^\eta f(x, y, t) = -\eta(I, -I, 1)w_{\varepsilon\mu}(t + \sigma) \\ & w_\eta(-t - \sigma + |x| + |y|)f(y, x, -t + |x| + |y|) \end{aligned}$$

and

$$(2.16) \quad \begin{aligned} & (T_3 K^{-1} (\mathcal{F}_1^{-1})^2 (\mathcal{F}_2^{-1})^2 \mathcal{F}_3^{-1} M_3 K^{-1} \mathcal{F}_3 K^{-1})^\eta f(x, y, t) \\ & = \eta(I, -I, 1)w_\eta(-t + |x| + |y|)w_\mu(t)f(y, x, -t + |x| + |y|). \end{aligned}$$

From (2.11)(c)

$$(2.17) \quad \begin{aligned} & (B_3(\chi)^{-1} \mathcal{F}_3^{-1} K^{-1} \mathcal{F}_3 B_3(\chi) T_3)^\eta f(x, y, t) \\ & = \eta(I, -I, 1)w_\mu(t + \sigma)w_\eta(-t - \sigma + |x| + |y|)f(y, x, -t + |x| + |y|) \end{aligned}$$

and

$$(2.18) \quad \begin{aligned} & (T_3 K^{-1} (\mathcal{F}_1^{-1})^2 (\mathcal{F}_2^{-1})^2 \mathcal{F}_3^{-1} K^{-1} \mathcal{F}_3 K^{-1})^\eta f(x, y, t) \\ & = \eta(I, -I, 1)w_\mu(t)w_\eta(-t + |x| + |y|)f(y, x, -t + |x| + |y|). \end{aligned}$$

Applying Lemma 2.9 and the formulas (2.13) and (2.14), we find that (2.11)(a) holds, and hence I_a commutes with $T(p)$, if and only if

$$(2.19) \quad \begin{aligned} & -\eta(I, -I, 1)w_{e\mu}(t)w_{\mu}(-t) \\ & = \eta(-I, I, 1)w_{e\eta}(\sigma - t - |x| - |y|)w_{\mu}(t + |x| + |y| - \sigma) \end{aligned}$$

for all $\eta \in \hat{G}$ and almost every $(x, y, t) \in Z$. In fact, since the two sides of (2.19) are continuous functions on Z , this equation must hold for every $(x, y, t) \in Z$. In particular, let $\eta = \theta$ be the trivial character, $t = 0$, and $|x| + |y| = \sigma$; then (2.19) becomes $-i = -w_e(0)w_{\theta}(0) = w_e(0)w_{\theta}(0) = i$, which is impossible. Thus I_a does not commute with $T(p)$ regardless of the choice of the unitary character χ of H .

Similarly, (2.11)(b) holds, and hence J_a commutes with $T(p)$, if and only if

$$(2.20) \quad \begin{aligned} & -\eta(I, -I, 1)w_{e\mu}(t + \sigma)w_{\eta}(-t - \sigma + |x| + |y|) \\ & = \eta(I, -I, 1)w_{\mu}(t)w_{e\eta}(-t + |x| + |y|) \end{aligned}$$

for every $(x, y, t) \in Z$ and every $\eta \in \hat{G}$. In particular, with $\eta = \theta$, the trivial character, $t = 0$, and $|x| + |y| = \sigma$, (2.20) becomes $-w_e(\sigma) = -w_e(\sigma)w_{\theta}(0) = w_e(\sigma)w_{\theta}(0) = w_e(\sigma)$ which is impossible since $w_e(\sigma)$ is never zero. Thus, J_a does not commute with $T(p)$ regardless of the choice of χ .

Finally, (2.11)(c) holds, and hence J commutes with $T(p)$, if and only if

$$(2.21) \quad w_{\mu}(t + \sigma)w_{\eta}(-t - \sigma + |x| + |y|) = w_{\mu}(t)w_{\eta}(-t + |x| + |y|),$$

for every $(x, y, t) \in Z$ and every $\eta \in \hat{G}$. If $\sigma = 0$ then (2.21) obviously holds. Thus, J commutes with $T(p)$ in case χ is the trivial character of H . We will now show that the assumption of (2.21) implies that $\sigma = 0$, so that J does not commute with $T(p)$ in any other case.

Let η , and hence μ , be the trivial character of G . Let $\gamma(t) = \Gamma(it/2 + 1/4)$; then (2.21) implies

$$(2.22) \quad \frac{\gamma(t + \sigma)\gamma(-t - \sigma + |x| + |y|)}{\gamma(-t - \sigma)\gamma(t + \sigma - |x| - |y|)} = \frac{\gamma(-t + |x| + |y|)\gamma(t)}{\gamma(t - |x| - |y|)\gamma(-t)}$$

for every $(x, y, t) \in Z$. Upon rearrangement, this becomes

$$(2.23) \quad \frac{\gamma(t + \sigma)\gamma(-t)}{\gamma(t)\gamma(-t - \sigma)} = \frac{\gamma(t + \sigma - |x| - |y|)}{\gamma(t - |x| - |y|)} \cdot \frac{\gamma(-t + |x| + |y|)}{\gamma(-t - \sigma + |x| + |y|)},$$

and uniqueness of Haar measure on $(\mathbf{R}, +)$ shows that

$$(2.24) \quad \frac{\gamma(t + \sigma)}{\gamma(t)} \cdot \frac{\gamma(-t)}{\gamma(-t - \sigma)} = \frac{\gamma(\sigma)}{\gamma(-\sigma)} \quad (\text{a constant}).$$

Differentiation of this equation with respect to t at $t = 0$ results in

$$(2.25) \quad \gamma'(\sigma)/\gamma(\sigma) + \gamma'(-\sigma)/\gamma(-\sigma) = 2\gamma'(0)/\gamma(0).$$

In terms of the Gamma function, since $\gamma'(\sigma) = i\Gamma'(i\sigma/2 + 1/4)/2$, this becomes

$$(2.26) \quad \frac{\Gamma'(i\sigma/2 + 1/4)}{\Gamma(i\sigma/2 + 1/4)} + \frac{\Gamma'(-i\sigma/2 + 1/4)}{\Gamma(-i\sigma/2 + 1/4)} = 2 \frac{\Gamma'(1/4)}{\Gamma(1/4)}.$$

Let $\psi(z) = \Gamma'(z)/\Gamma(z)$, and set $z = i\sigma/2 + 1/4$. Then (2.26) can be rewritten as

$$(2.27) \quad \text{Re } \psi(z) = \psi(\text{Re } z).$$

The next lemma, which follows from a well-known expansion for ψ [1, p. 189], finishes the proof.

LEMMA 2.28. *Let $\psi(z) = \Gamma'(z)/\Gamma(z)$, with $z \in \mathbb{C}$ and $\operatorname{Re} z > 0$. Then $\operatorname{Re} \psi(z) = \psi(\operatorname{Re} z)$ if and only if $\operatorname{Im} z = 0$.*

We have proved that J commutes with $T(p)$ if and only if χ is the trivial character.

We give a formal statement of the results we have just proved. To this end recall that $T = T_\chi$, and let $P_\pm = (I \pm J)/2$ and $L^2(Z)^\pm = P_\pm(L^2(Z))$.

THEOREM 2.29. *Let χ be a unitary character of $\bar{H}$ and T be the degenerate principal series representation of $\operatorname{Sp}(n, \mathbb{R})$ induced by χ and acting in $L^2(Z)$. If χ is not the trivial character then T is irreducible. If χ is the trivial character then $J \in \mathcal{Q}'(T)$, the subrepresentations $T^\pm$ of T on $L^2(Z)^\pm$ are irreducible, and $T = T^+ \oplus T^-$.*

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