ON THE DEPTH OF LOCAL RINGS OF INVARIANTS OF CYCLIC GROUPS

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ABSTRACT. For wild actions of a cyclic group G on a local ring R , with R^G noetherian, it is shown that $\text{depth } R - \text{depth } R^G$ can be arbitrarily large, even if R is regular and contains $\mathbb{Z}$.

1. Introduction. Let R be a noetherian local ring and let G be a finite group of automorphisms of R . If the ring R^G of G -invariants in R is noetherian, what is the relation between the depth of R and the depth of R^G ? (For the problem of when R^G is noetherian, see [1].) This question has been raised in connection with problems of unique factorization (see [4, esp. §4]). The theme of this note is that when the residue field characteristic of R divides the order of G , things are as bad as possible, at least if one expects that the depth of R will govern the depth of R^G . We exhibit a class of actions of a cyclic group G of order p^n in characteristic p for which the depth of R^G is 2, provided the depth of R is at least 2. There is also a related phenomenon in mixed characteristic where the ring of invariants always has $\text{depth} \leq 3$.

2. The depth of R^G . Recall that if R is a noetherian local ring with maximal ideal $\mathfrak{m}$, the depth of R is the largest integer $d \geq 0$ for which there exists a sequence $f_0, f_1, \dots, f_d$ of elements of $\mathfrak{m}$ with $f_0 = 0$ and $((f_0, \dots, f_{i-1}) : (f_i)) = (f_0, \dots, f_{i-1})$, $1 \leq i \leq d$.

Let G be a finite group of automorphisms of R such that R^G is noetherian. Let g be the order of G .

PROPOSITION 1. *If g is invertible in R , then $\text{depth}(R^G) \geq \text{depth}(R)$.*

PROOF. If $\text{depth}(R) = 0$, there is nothing to prove, so assume that $\text{depth}(R) > 0$ and let $f \in \mathfrak{m}$ be a non-zero-divisor. Then $F = \text{Norm}(f)$ is a non-zero-divisor in R^G and $(FR)^G = FR^G$. Now $\text{depth}(R/FR) = \text{depth}(R) - 1$ and $\text{depth}(RG/FR^G) = \text{depth}(R^G) - 1$. Since g is invertible in R , $(R/FR)^G = R^G/FR^G$ and the proposition follows by induction on the depth.

That the inequality in Proposition 1 may be strict is illustrated by the following simple example. Let k be the complex field and let $R = k[[X, Y]]/(XY, Y^2)$. Let G be the group generated by the automorphism, $X \mapsto X, Y \mapsto -Y$, of R . R has depth 0 and $R^G = k[[X]]$ has depth 1.

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Since R is integral over R^G , one always has $\dim(R^G) = \dim(R)$, so that if g is invertible in R , then R Cohen-Macaulay $\Rightarrow R^G$ Cohen-Macaulay. In the case where g may not be invertible in R , all that can be said in general is

PROPOSITION 2. *Let d be one of 0, 1 or 2. If $\text{depth}(R) \geq d$, then $\text{depth}(R^G) \geq d$.*

PROOF. The case $d = 0$ is obvious. If $d = 1$, let f be a non-zero-divisor in the maximal ideal of R . Then, as before, $F = \text{Norm}(f)$ is a non-zero-divisor in R^G and $(FR)^G = FR^G$. If $d = 2$, then $\text{depth}(R/FR) \geq 1$. If h is a non-zero-divisor in the maximal ideal of $(R/FR)^G$, then h^g is a non-zero-divisor in R^G/FR^G . Hence there is an H in the maximal ideal of R^G such that F, H is an R^G -sequence.

We show below that for every $d \geq 2$, there is an R of depth d with R^G of depth 2. For this, we need the following version of the normal basis theorem.

LEMMA 3. *Let S be a noetherian semilocal domain and let G be a finite group of automorphisms of S operating transitively on the maximal ideals of S . If S is a finite étale extension of $R = S^G$ then there exists an $f \in S$ such that $\{f^\sigma: \sigma \in G\}$ is a free basis of S over R .*

PROOF. Since G acts transitively on the maximal ideals of S and S is faithfully flat over R , it follows that R is a noetherian local ring. Let $\mathfrak{m}$ be its maximal ideal. Then $S/\mathfrak{m}S$ is a direct sum of finite separable extensions $K_i = S/\mathfrak{n}_i$ of $R/\mathfrak{m} = k$, $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ being the maximal ideals of S . Set $G_j = \{\sigma \in G: \sigma \equiv 1 \pmod{\mathfrak{m}'S}\}$. Then $G_n = (1)$ for large n . Fix such an n . If G_1 contains a subgroup H whose order k is invertible in $S/\mathfrak{m}S$, then since $(S/\mathfrak{m}^nS)^H$ maps onto $S/\mathfrak{m}S$, and $S/\mathfrak{m}^nS$ is a finite $R/\mathfrak{m}^n$ -module, it follows from Nakayama's lemma that $(S/\mathfrak{m}^nS)^H = S/\mathfrak{m}^nS$, so that $H = (1)$. Thus if k has characteristic 0, then $G_1 = (1)$ and if k has characteristic $p > 0$, then G_1 is a p -group. In the latter case G_1 is still trivial. To see this, note that G_1 permutes transitively the maximal ideals of S over a given maximal ideal of S^{G_1} , for if $\mathfrak{n}_i^{G_1} = \mathfrak{n}_j^{G_1}$ and $f \in \mathfrak{n}_i$ then $\prod_{\sigma \in G_1} f^\sigma \in \mathfrak{n}_j$ so $f^t \in \mathfrak{n}_j$ for some $t \in G_1$, i.e., $\mathfrak{n}_i \subset \bigcup_{\sigma \in G_1} \mathfrak{n}_j^\sigma$, and so $\mathfrak{n}_i = \mathfrak{n}_j^\sigma$ for some $\sigma \in G_1$. As G_1 acts trivially on $S/\mathfrak{m}S$, there is a unique maximal ideal of S over each maximal ideal of S^{G_1} . If G_1 has order $q = p^a$, then for any $f \in K_i$, $f^q \in S^{G_1}/\mathfrak{n}_i^{G_1}$, so that $S^{G_1}/\mathfrak{n}_i^{G_1} = K_i$ by separability. Thus the canonical map $S^{G_1}/\mathfrak{m}S^{G_1} \rightarrow S/\mathfrak{m}S$ is surjective. By Nakayama's lemma $S^{G_1} = S$ and $G_1 = (1)$. Therefore G acts faithfully on $S/\mathfrak{m}S$.

Let K be the fraction field of S . Then K^G is the fraction field of R and $K = K^G \otimes_R S_1$ which shows that S is free of rank $g = \text{order of } G \text{ over } R$. Hence $[S/\mathfrak{m}S: k] = g$. Since G is transitive on the K_i , $K_0 = (S/\mathfrak{m}S)^G$ is an extension field of k . If this is the stabilizer of K_i in G , then K_0 maps onto $K_i^{H_i}$. Let h be the common order of the conjugate subgroups H_i of G . Then $[S/\mathfrak{m}S: K_0] = rh = g$, so $K_0 = k$. Choose $\delta \in K$, so that $\{\delta^\sigma: \sigma \in H_1\}$ is a basis of K_1 over k . Let $f \in S$ map to $\delta \pmod{\mathfrak{m}S}$. Then $\{\delta^\sigma: \sigma \in G\}$ is a k -basis of $S/\mathfrak{m}S$ and so, by Nakayama's lemma, $\{f^\sigma: \sigma \in G\}$ is a free basis of S over R .

Now let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $X = \text{Spec}(R)$ and let $\dot{X} = X - \{\mathfrak{m}\}$ be the punctured spectrum of R . Recall that $\text{depth}(R) = \{q: H_{\mathfrak{m}}^q(R) \neq (0)\}$, where $H_{\mathfrak{m}}^q$ denotes the local cohomology (cf. [3]). One also has

an exact sequence

$$0 \rightarrow H^0(R) \rightarrow R \rightarrow H^0(\dot{X}, \mathcal{O}_X) \rightarrow H^1(R) \rightarrow 0$$

and isomorphisms $H^q(\dot{X}, \mathcal{O}_X) \xrightarrow{\sim} H_m^{q+1}(R)$ for $q > 0$. Thus $\text{depth}(R) < \sup(2, q)$ whenever $H^{q-1}(\dot{X}, \mathcal{O}_X) \neq (0)$.

If Y is an affine scheme and G is a finite group acting faithfully on Y , we say that G acts freely on Y if the map $Y \rightarrow \text{Spec}(H^0(Y, \mathcal{O}_Y)^G)$ is étale. For an arbitrary scheme Y we say that G acts freely on Y if there is a cover of Y by G -stable affine opens Y_i such that G acts freely on each Y_i .

PROPOSITION 4. *Let R be a noetherian local domain with maximal ideal $\mathfrak{m}$ and let $X = \text{Spec}(R)$. Let $p > 0$ be the characteristic of $R/\mathfrak{m}$ and let G be a cyclic group of order $p^n h$, $n > 0$, acting on R such that*

- (i) R^G is noetherian and R is a finite R^G -module;
- (ii) G operates trivially on $R/\mathfrak{m}$;
- (iii) G acts freely on $\dot{X}$.

Then

- (a) if R is equicharacteristic of depth > 2 , then $\text{depth}(R^G) < 2$;
- (b) if $\text{depth}(R) \geq 3$, then $\text{depth}(R^G) < 3$.

REMARK. The hypothesis (i) will be satisfied if the module $\Omega_{(R/pR)/\mathbb{Z}}$ of Kähler differentials of R/pR is a finite R -module (see [1]).

To prove the proposition, we first show that, in case (a), $H^1(G, R) \neq (0)$, and in case (b), $H^2(G, R) \neq (0)$. In either case if σ is a generator of G , we have

$$H^1(G, R) = (\text{elements of trace } 0) / (\text{all } f^\sigma - f, f \in R),$$

$$H^2(G, R) = R^G / (\text{traces}).$$

Since G operates trivially on $R/\mathfrak{m}$, we have $f^\sigma - f \in \mathfrak{m}$ for all $f \in R$, and so $1 \neq f^\sigma - f$ for any f . But in case (a), $\text{trace}(1) = 0$. In addition, 1 is not a trace, for if $1 = \text{trace}(f)$, then in $R/\mathfrak{m}$ we have $1 = p^n h \bar{f} = 0$, where $\bar{f}$ is the residue of f .

Let $Y = \text{Spec}(R^G)$. Then $\dot{Y}$ is a geometric quotient of $\dot{X}$ by the action of G (see [5]). There are two spectral sequences with the same abutment, whose $E_2^{p,q}$ terms are

$$\text{I: } H^q(G, H^p(\dot{X}, \mathcal{O}_X)),$$

$$\text{II: } H^p(\dot{Y}, \underline{H}^q(G, \mathcal{O}_X)),$$

where $\underline{H}^q(G, \mathcal{O}_X)$ is the sheaf associated to the presheaf on $\dot{Y}$, $\mathcal{U} \mapsto H^q(G, H^0(V, \mathcal{O}_X))$, V being the inverse image of $\mathcal{U}$. By Lemma 3, $\underline{H}^q(G, \mathcal{O}_X) = (0)$, $q > 0$, and II collapses. Thus we have a single spectral sequence $H^q(G, H^p(\dot{X}, \mathcal{O}_X)) \Rightarrow H^*(\dot{Y}, \mathcal{O}_Y)$. Since $\text{depth}(R) \geq 2$, it follows that $H^0(\dot{X}, \mathcal{O}_X) = R$ and the exact sequence of terms of low degree is

$$0 \rightarrow H^1(G, R) \rightarrow H^1(\dot{Y}, \mathcal{O}_Y) \rightarrow H^1(\dot{X}, \mathcal{O}_X)^G \rightarrow H^2(G, R) \rightarrow H^2(\dot{Y}, \mathcal{O}_Y).$$

In case (a), $H^1(G, R) \neq (0)$, so $H^1(\dot{Y}, \mathcal{O}_Y) \neq (0)$, and $\text{depth}(R^G) > 2$. In case (b), $H^1(\dot{X}, \mathcal{O}_X) = (0)$ since $\text{depth}(R) \geq 3$. But $H^2(G, R) \neq (0)$, so $H^2(\dot{Y}, \mathcal{O}_Y) \neq (0)$ and $\text{depth}(R^G) < 3$. Q.E.D.

Combining case (a) with Proposition 2 shows that one always has $\text{depth}(R^G) = 2$, provided $\text{depth}(R) \geq 2$.

3. An example. We give an example which shows that the inequality of case (b) in Proposition 4 cannot be improved. Let A be the ring of Gaussian integers and let R_0 be the localization of A at the prime $(1 + i)A$. Complex conjugation induces an automorphism of R_0 . The residue field of R_0 is $\mathbb{F}_2$. Let $S = R_0[X_1, \dots, X_n, Y_1, \dots, Y_n]$ be the ring of polynomials over R_0 in the two sets of variables $\underline{X} = (X_1, \dots, X_n)$, $\underline{Y} = (Y_1, \dots, Y_n)$ and let the group G of order 2 operate on S as follows. If $f \in S$, write $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$ —in the usual notation, $\alpha = (\alpha_1, \dots, \alpha_n)$, etc. Let $\bar{f} = \sum \bar{a}_{\alpha\beta} Y^\alpha X^\beta$. Let $V = \text{Spec}(S)$, $V_0 = \text{Spec}(R_0)$. If x is the closed point and ζ the generic point of V_0 , then $V_x = \text{Spec}(\mathbb{F}_2[\underline{X}, \underline{Y}])$, and $V_\zeta = \text{Spec}(\mathbb{Q}(i)[\underline{X}, \underline{Y}])$. Also $V^G \subset V_x^G$, for if y is a point of $V - V_x$, then $\mathbb{Q}(i)$ is contained in the residue field $\kappa(y)$ at y , so if G leaves y fixed, it operates nontrivially in $\kappa(y)$. Thus V^G is defined by the ideal

$$I = (2, X_1 - Y_1, \dots, X_n - Y_n),$$

in S . Let $P = \sqrt{I}$. Then P is prime in S . Let $R = S_P$. Then the residue field of R is the function field of V^G , upon which G operates trivially.

Let $\bar{S} = S/(1 + i)S \approx \mathbb{F}_2[\underline{X}, \underline{Y}]$. Every invariant in $\bar{S}$ is a sum of elements of the form

$$\begin{aligned} \underline{X}^\alpha \underline{Y}^\beta + \underline{X}^\beta \underline{Y}^\alpha, \quad \alpha \neq \beta, \\ \underline{X}^\alpha \underline{Y}^\alpha. \end{aligned}$$

On the other hand, every invariant in S is a sum of elements of the form

$$\begin{aligned} a \underline{X}^\alpha \underline{Y}^\beta + \bar{a} \underline{X}^\beta \underline{Y}^\alpha, \quad \alpha \neq \beta, a \in R_0, \\ b \underline{X}^\alpha \underline{Y}^\alpha, \quad b \in R_0 \cap \mathbb{Q}. \end{aligned}$$

Thus $\bar{S}^G = S^G/((1 + i)S)^G$, and so if R denotes $\bar{R}/(1 + i)R$, then $\bar{R}^G = R^G/((1 + i)R)^G$.

We show now that 2 is a prime element of R^G . For this it suffices to show that 2 is prime in S^G , i.e., if $f \in S$ is such that $(1 + i)f \in S^G$, then $(1 + i)f = 2g$, $g \in S^G$. (Since $(1 + i)$ is prime in R_0 , it is prime in S .) However $(1 + i)f \in S^G$ implies $\bar{f} = \bar{if}$, i.e., the coefficients $a_{\alpha\beta}$ of f satisfy $\bar{a}_{\alpha\beta} = ia_{\beta\alpha}$ and we get $i(a_{\alpha\beta} \pm a_{\beta\alpha}) = \overline{a_{\alpha\beta} \pm a_{\beta\alpha}}$, $\alpha \neq \beta$, and $ia_{\alpha\alpha} = \overline{a_{\alpha\alpha}}$. Thus $a_{\alpha\beta} = c_{\alpha\beta}(1 - i)$, $c_{\alpha\beta} \in R_0 \cap \mathbb{Q}$. Also

$$c_{\alpha\beta}(1 + i) = ic_{\alpha\beta}(1 - i) = \overline{c_{\beta\alpha}(1 - i)} = c_{\beta\alpha}(1 + i),$$

so $c_{\alpha\beta} = c_{\beta\alpha}$. Let $g = \sum c_{\alpha\beta} \underline{X}^\alpha \underline{Y}^\beta$. Then $g \in S^G$ and $f = (1 - i)g$.

Therefore $R^G/2R^G = \bar{R}^G$. R (resp., $\bar{R}$) is a regular local ring of dimension $n + 1$ (resp., n). If the hypotheses of Proposition 4 are satisfied for R , they are also satisfied for $\bar{R}$, and to prove them for R , it suffices to show that the quotient map $\phi: V \rightarrow V/G = \text{Spec}(R^G)$ is étale on $V - V^G$. If K is the fraction field of S , then K^G is the fraction field of S^G and $K^G \otimes_{S^G} S = K$. Also, for all closed points $z \in \phi(V - V^G)$ we have

$$\dim_{\kappa(z)} S \otimes_{S^G} \kappa(z) = [K : K^G] = 2,$$

from which it follows that $\phi|V - V^G$ is flat. Since the residue fields of closed points of V/G are either finite or of characteristic 0, it follows that all such residue field extensions are separable. Finally, $\phi(V - V^G)$ is exactly the set over which ϕ is unramified. Therefore, taking $n > 2$ and applying case (a) to $\bar{R}$, we get $\text{depth}(R^G) = 3$.

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