HEREDITARY FINITELY GENERATED ALGEBRAS SATISFYING A POLYNOMIAL IDENTITY

ELLEN E. KIRKMAN AND JAMES KUZMANOVICH

ABSTRACT. If Λ is a right and left p.p. ring which satisfies a polynomial identity and is a finitely generated algebra over its center, then $\Lambda \simeq \Gamma \times \Omega$, where Γ is a semiprime ring having a von Neumann regular classical quotient ring which is module-finite over its center and Ω has nonzero prime radical at each of its Pierce stalks. If Λ is right and left hereditary, then Γ is an order over a commutative hereditary ring in the sense of [7]; the ring Ω is then a direct product of finitely many indecomposable piecewise domains.

Robson and Small [10] showed that hereditary prime PI rings are classical hereditary orders; in this paper we generalize their result to rings which are hereditary, semiprime, and finitely generated over their centers. Jøndrup [6] proved that a semiprime PI ring which is module-finite over its center and a right hereditary ring must also be a left hereditary ring; our Theorem 5 extends this result. Jøndrup [5] showed that a semiprime, right p.p., finitely generated PI algebra has a right classical quotient ring; our Corollary 4 strengthens this result, and an example is given to show that the finitely generated hypothesis cannot be dropped.

A major tool used in this paper is the Pierce sheaf representation of a ring Λ , and we now introduce the sheaf representation terminology which is used throughout. Complete details can be found in Pierce [8]. Let the Boolean ring of central idempotents of Λ be denoted by $B(\Lambda)$ and let $X(\Lambda)$ (or just X if there is no chance of confusion) denote $\text{Spec}(B(\Lambda))$. The space X is a totally disconnected, compact, Hausdorff space. Pierce's sheaf has X as its base space and has $\Lambda_x = \Lambda/x\Lambda$ as the stalk at x for $x \in X$. If $\lambda \in \Lambda$, then λ_x denotes the image of λ in the stalk Λ_x . The ring Λ is then isomorphic to the ring of global sections of this sheaf. If $\lambda \in \Lambda$, then $\text{sup}(\lambda) = \{x \in X: \lambda_x \neq 0_x\}$ is called the support of λ . For an open-closed subset V of X , the central idempotent whose support is V is called the associated idempotent of V .

Let R be a commutative ring with total quotient ring K and let Σ be a central separable K -algebra. A module-finite R -subalgebra Λ of Σ such that $\Lambda K = \Sigma$ is called an R -order in Σ . The structure of a hereditary R -order Λ over a commutative hereditary ring Λ was described in [7].

The following lemma gives information about the Pierce stalks of Λ when Λ is semiprime and only right p.p.

Received by the editors October 22, 1980.

1980 *Mathematics Subject Classification*. Primary 16A14, 16A08, 16A38; Secondary 16A16.

© 1981 American Mathematical Society
0002-9939/81/0000-0505/\$02.50

LEMMA 1. Let Λ be a PI ring which is finitely generated as an algebra over its center R . Suppose that Λ is right p.p. If $x \in X(\Lambda)$ and if J is a nilpotent right ideal of Λ_x , then J can be lifted to a nilpotent right ideal I of Λ . Hence, if Λ is semiprime, then each Λ_x is prime.

PROOF. By [9, Chapter V, Theorem 2.5, p. 108] $\Lambda_x/N(\Lambda_x)$ is a semiprime Goldie ring, where $N(\Lambda_x)$ is the prime radical of Λ_x , and thus Λ_x has no infinite family of orthogonal idempotents. Since Λ is right p.p, so is Λ_x ; thus Λ_x is a PWD (piecewise domain) [3]. By [3], Λ_x has a triangular structure given by

$$\Lambda_x \simeq \begin{bmatrix} P_1 & & & & \\ & P_2 & & & \\ & P_{21} & P_2 & & \\ & \vdots & & \ddots & \\ & P_{m1} & P_{m2} & \cdots & P_m \end{bmatrix}$$

where each diagonal ring P_i is prime and where $N(\Lambda_x)$ is represented by the P_{ij} blocks below the diagonal. Let $\{f_1, \dots, f_m\}$ be a family of orthogonal idempotents of Λ such that $(f_i)_x \Lambda_x (f_i)_x = P_i$ and $(f_i)_x \Lambda_x (f_j)_x = P_{ij}$. Let $\{g_1, \dots, g_n\}$ generate Λ as an algebra over R . Due to the triangular structure of Λ_x , we have

$$(*) \quad (g_k)_x = \sum_{\substack{i=1 \\ j < i}}^m (f_i)_x (g_k)_x (f_j)_x$$

for $k = 1, \dots, n$. These equations must then hold in an open-closed neighborhood U of x . Let e be the associated central idempotent of U . Since the equations $(*)$ hold for each $y \in U$, and since $\{g_1, \dots, g_n\}$ generates Λ over R , it follows that $e\Lambda$ has a triangular structure given by

$$e\Lambda \simeq \begin{bmatrix} (ef_1)e\Lambda(ef_1) & & & 0 \\ & \vdots & & \\ & & \ddots & \\ (ef_m)e\Lambda(ef_1) & \cdots & & (ef_n)e\Lambda(ef_n) \end{bmatrix}.$$

Let

$$M = e \left[\sum_{\substack{i=2 \\ j < i}}^m f_i \Lambda f_j \right] = \sum_{\substack{i=2 \\ j < i}}^m (ef_i)e\Lambda(ef_j).$$

It is clear from the triangular structure of $e\Lambda$ that M is an ideal of Λ which is nilpotent of index at most m . Let $I = \{\lambda \in M: \lambda_x \in J\}$. It is easily verified that I is a nilpotent right ideal for which $I_x = J$.

The ideal M is a lifting of $N(\Lambda_x)$ to a nilpotent ideal of Λ ; it follows that if Λ is semiprime, then Λ_x is semiprime. The ring Λ_x is thus prime, for a semiprime PWD is a direct sum of prime rings, but Λ_x must be indecomposable since Λ is a finitely generated algebra over R . $\square$

Bergman [1] characterized the commutative p.p. rings as being those commutative rings whose Pierce stalks were all integral domains and in which elements had open-closed support. The following is a generalization of his result. The two-sided p.p. assumption is needed in the hypothesis.

PROPOSITION 2. *Let Λ be a two-sided p.p. ring which is PI and which is a finitely generated algebra over its center R . The elements of Λ have open-closed support.*

PROOF. Let $\lambda \in \Lambda$. Since Λ is p.p. there are idempotents e and f in Λ such that $\text{rann}(\lambda) = \text{rann}(e)$ and $\text{lann}(\lambda) = \text{lann}(f)$. It is not difficult to see that $\text{sup}(\lambda) = \text{sup}(e) = \text{sup}(f)$ and $f\lambda = \lambda = \lambda e$.

Take $0 \neq \sigma \in \Lambda$. Supports are always closed; therefore, it will be sufficient to show that $\text{sup}(\sigma)$ is open. Let $x \in X(\Lambda)$ be an element of $\text{sup}(\sigma)$. Define $I = \{u \in \Lambda_x : u = \lambda_x \text{ for } \lambda \in \Lambda \text{ with } \text{sup}(\lambda) \subset \text{sup}(\sigma)\}$. The ideal I is a two-sided ideal of Λ_x , and since $x \in \text{sup}(\sigma)$, I is nonzero. By the above observation, for every $u \in I$ with $u = \lambda_x$, there exist idempotents $e = e(\lambda)$, $f = f(\lambda)$ in Λ such that $f\lambda = \lambda = \lambda e$ and $e_x \in I$, $f_x \in I$. Thus if J is any ideal of Λ_x , then $I \cap J = IJ = JI$.

Let N be the prime radical of Λ_x . As in Lemma 1, $\bar{\Lambda}_x = \Lambda_x/N$ (use bars to denote images in $\bar{\Lambda}_x$) is a semiprime Goldie ring; $\bar{\Lambda}_x$ is also p.p. so that it is a direct product of finitely many prime rings. Let S be one such prime factor in which I has nonzero image $\bar{I}$. In this case $\bar{I}$ contains a regular element $\bar{b}$ of S , since S is a prime Goldie ring. Let e be an idempotent in I such that $eb = b$. Clearly $\bar{e}\bar{b} = \bar{b}$, which by the regularity of $\bar{b}$ implies that $\bar{e} = 1_S$ and $\bar{I} = S$. This holds for each such prime factor of $\bar{\Lambda}_x$ from which it follows that there is an idempotent h in I which is central in $\bar{\Lambda}_x$ for which $I = h\Lambda_x + (I \cap N)$. Since N is nilpotent, $I = h\Lambda_x + (I \cap N) = h\Lambda_x + IN = h\Lambda_x$. Similarly, there is an idempotent g in I for which $I = \Lambda_x g$. It then follows that $h = hg = g$ is a central idempotent of Λ_x . Since Λ is a finitely generated algebra, Λ_x is indecomposable and $h = 1_x$. Lift h to an idempotent f of Λ such that $f_x = h = 1_x$ and $\text{sup}(f) \subset \text{sup}(\sigma)$. There is an open-closed neighborhood of x with associated central idempotent k such that $1_y = f_y$ for all y in the neighborhood. Then fk is a central idempotent of Λ with $x \in \text{sup}(fk) \subset \text{sup}(f) \subset \text{sup}(\sigma)$. Thus $\text{sup}(fk)$ is an open neighborhood of x contained in $\text{sup}(\sigma)$. Consequently $\text{sup}(\sigma)$ is open as desired. $\square$

The following theorem is a generalization of results in [2].

THEOREM 3. *Let Λ be a PI ring which is a finitely generated algebra over its center R .*

(i) *If Λ is left and right p.p., then $\Lambda = \Gamma \times \Omega$ where Γ is a semiprime ring while the ring Ω has nonzero prime radical at each of its Pierce stalks. The ring Γ has a regular total quotient ring which is module-finite over its center and is obtained from Γ by inverting regular central elements.*

(ii) *If in addition Λ is hereditary, then the semiprime ring Γ is module-finite over its center (and hence its center is hereditary and it is an order in the sense of [7]) and the ring Ω is the direct product of finitely many indecomposable piecewise domains with nonzero prime radical.*

PROOF. (i) It is sufficient to show that the set $S = \{x \in X: \Lambda_x \text{ is prime}\}$ is open-closed. Let Σ be the ring obtained from Λ by inverting the regular central elements. It is easy to see that $X(\Lambda) = X(\Sigma)$ and that K is the center of Σ where K is the total quotient ring of R .

Take $x \in S$. The ring Λ_x is a prime PI ring; consequently, an extension of Posner's theorem [9, p. 175] implies that Σ_x is the simple Artinian classical quotient ring of Λ_x . Hence $S = \{x \in X: \Sigma_x \text{ is a central separable } K_x\text{-algebra}\}$. For a finitely generated algebra the property of being module-finite over its center can be expressed in a finite number of equations. Therefore, if $x \in S$, then there exists an open-closed neighborhood U of x such that Σ_y is module-finite over K_y for all $y \in U$. The separability of a module-finite algebra can also be expressed in terms of a finite number of equations; hence there is an open-closed neighborhood $V \subset U$ of x for which Σ_y is central separable for all $y \in V$. This shows that S is an open set.

Conversely, suppose that $x \notin S$; that is, suppose that Λ_x is not prime (and hence not semiprime as noted above). Let J be a nonzero nilpotent ideal of Λ_x . By Lemma 1 there exists a nilpotent ideal I of Λ such that $I_x = J$. Applying Proposition 2 yields that supports of elements are open-closed and hence that there exists an open-closed neighborhood U of x such that I_y is a nonzero nilpotent ideal of Λ_y for all $y \in U$. This shows that Λ_y is not prime for each such y and that $U \subset X - S$. Hence $X - S$ is open and S is closed.

Let e be the central idempotent of Λ for which $\Gamma = e\Lambda$. The ring $e\Sigma$ is regular since $(e\Sigma)_x$ is simple Artinian for each $x \in X(\Gamma)$. It follows that $e\Sigma$ is the total quotient ring of Γ , for it is regular and is obtained by inverting regular central elements of Λ . A standard sheaf argument shows that $e\Sigma$ is module-finite over its center.

(ii) If $x \in S = X(\Gamma)$, then Γ_x is a hereditary prime PI ring if Λ is hereditary. Hence by [10] each Γ_x is a classical order over its center which is a Dedekind domain. The property of being module-finite over its center can be expressed in finitely many equations for a finitely generated algebra. A standard sheaf argument then yields that Γ is module-finite over its center.

To show that Ω is a direct product of finitely many indecomposable rings it will be sufficient to show that $X(\Omega)$ is finite. Suppose that $X(\Omega)$ is infinite. In this case $X(\Omega)$ has a limit point y since $X(\Omega)$ is compact. Let $M = \{w \in \Omega: w_y = 0_y\}$. Let $u \in \Omega$ such that $u_y \neq 0_y$ and $(u_y \Omega_y)^2 = 0_y$. Let $I = M + u\Omega$. Observe that since $X(\Omega)$ is Hausdorff, $I_x = \Omega_x$ if $x \neq y$ and $I_y = u_y \Omega_y$. A contradiction will be obtained by showing that the ideal I cannot be projective. If I were projective, then $I = \bigoplus J_\alpha$ where each J_α is a finitely generated right ideal Ω . There is no loss of generality in assuming that exactly one J_α , say J_1 , is nonzero at y ; hence $(J_1)_y = (u_y \Omega_y)$. By Lemma 1 $(u_y \Omega_y)$ can be lifted to a nilpotent right ideal of Ω . The right ideal J_1 is finitely generated, so there is an open-closed neighborhood U of y such that $(J_1)_z$ is nilpotent for all $z \in U$. By Proposition 2 supports are open-closed so there is no loss of generality in assuming that $(J_1)_z \neq 0_z$ for all $z \in U$. Since y is a limit point, there exists $x \in U$, $x \neq y$. The ideal $I_x = (\bigoplus J_\alpha)_x = \bigoplus (J_\alpha)_x = \Omega_x$.

The nilpotent right ideal $(J_1)_x$ is superfluous in Ω_x , which implies that $\bigoplus_{\alpha \neq 1} (J_\alpha)_x = \Omega_x$. However, $0_x \neq (J_1)_x = (J_1)_x \cap (\bigoplus_{\alpha \neq 1} (J_\alpha)_x)$. This contradicts the fact that the sum of the family $\{J_\alpha\}$ is direct. Hence I is not projective and $X(\Omega)$ is finite.

It was noted in the proof of Lemma 1 that each Ω_x is a PWD. $\square$

COROLLARY 4. *Let Λ be a semiprime PI ring which is a finitely generated algebra over its center R and let K be the total quotient ring of R . If Λ is right p.p., then Λ has a regular classical quotient ring Σ which is module-finite over its center K and is obtained from Λ by inverting regular central elements.*

PROOF. The proof of Theorem 3 uses the two-sided p.p. hypothesis only to obtain the ring decomposition; hence, if Λ is semiprime the assumption of right p.p. is sufficient to obtain the properties of Σ . $\square$

Theorem 3 can be interpreted as saying that a hereditary, finitely generated, PI algebra is at most a PWD ring direct summand away from being module-finite over its center. Indeed, such PWD's need not be module-finite algebras; an example of such is the ring

$$\Omega = \begin{bmatrix} F & 0 & 0 \\ F[x] & F[x] & 0 \\ F[x] & F[x] & F \end{bmatrix}.$$

The ring Ω above does not have a classical quotient ring which shows that the semiprime hypothesis cannot be dropped in Corollary 4.

The finitely generated hypothesis also cannot be dropped in Corollary 4. Let

$$\Gamma = \bigoplus \begin{bmatrix} F(x) & F(x) & F(x) \\ F(x) & F(x) & F(x) \\ F(x) & F(x) & F(x) \end{bmatrix} + \text{long} \begin{bmatrix} F & 0 & 0 \\ F[x] & F[x] & 0 \\ F[x] & F[x] & F \end{bmatrix}$$

where x is an indeterminant over the field F and "long" denotes a constant sequence of the indicated form. The ring Γ is a hereditary semiprime PI ring, but it does not possess a classical quotient ring. This answers a question of Jøndrup [5].

Finally, it was indicated in [2, Example 2.3] that the ring decomposition of Theorem 3 cannot be obtained under the hypothesis that Λ is hereditary on only one side, even when Λ is module-finite over R . This example [2, Example 2.3] also shows that the semiprime hypothesis is needed in the following theorem.

THEOREM 5. *Let Λ be a semiprime PI ring which is a finitely generated algebra over its center R .*

- (i) *If Λ is right p.p., then Λ is left p.p.*
- (ii) *If Λ is right semihereditary, then Λ is left semihereditary.*
- (iii) *If Λ is right hereditary, then Λ is left hereditary.*

PROOF. (i) Take $\lambda \in \Lambda$. Let Σ be the regular classical quotient ring of Λ which exists by Corollary 4. Each Λ_x is a prime Goldie ring and hence cannot contain an infinite family of orthogonal idempotents. It follows from a result of Small [11] that each Λ_x is left p.p.; hence, for each $x \in X$ there exists an idempotent $f(x)$ in Λ

such that $\text{lann}_{\Lambda_x}(\lambda_x) = \Lambda_x f(x)_x$. The ring Σ is regular, whence $\text{lann}_{\Sigma}(\lambda) = \Sigma e$ for an idempotent e of Σ . Since $e = b^{-1}a$ where $a \in \text{lann}_{\Lambda}(\lambda)$ and b is a regular central element of Λ , the equations

$$(**) \quad a_x f(x)_x = a_x, \quad f(x)_x \lambda_x = 0_x, \quad (f(x)_x)^2 = f(x)_x$$

hold for each $x \in X$. The equations (**) can be lifted and patched in the standard sheaf theoretic manner to yield an idempotent f of Λ for which $ef = e$ and $f\lambda = 0$. Clearly $\Lambda f \subset \text{lann}_{\Lambda}(\lambda)$. Conversely, let $\rho \in \text{lann}_{\Lambda}(\lambda)$. Since $\text{lann}_{\Lambda}(\lambda) \subset \text{lann}_{\Sigma}(\lambda) = \Sigma e$, we have $\rho f = \rho e f = \rho e = \rho$. Thus $\text{lann}_{\Lambda}(\lambda) = \Lambda f$ and Λf is projective. This shows that Λ is left p.p.

(ii) If Λ is right semihereditary, then $M_n(\Lambda)$, the $n \times n$ matrix ring over Λ , is right p.p. for any $n > 1$. All the hypotheses of (i) apply to $M_n(\Lambda)$, so that (i) implies that $M_n(\Lambda)$ is left p.p. for $n > 1$; that is, Λ is left semihereditary.

(iii) If Λ is right hereditary, then each Λ_x is a right hereditary prime PI ring, and by [10] each Λ_x is then module-finite over its center. The ring Λ is hereditary by [4]. Hence Λ is an order over a hereditary ring in the sense of [7]. By [7, Theorem 3.5] Λ is left hereditary. $\square$

As a corollary to Theorem 3 and Theorem 5 we have the following. It can be viewed as a generalization of Robson and Small [10].

COROLLARY 6. *Let Λ be a right hereditary semiprime PI ring which is a finitely generated algebra over its center. Then Λ is a hereditary order over a commutative hereditary ring.* $\square$

Theorem 5 is not valid for semiprime PI rings which are not finitely generated algebras over their centers. For a field F denote elements of $T = (\Pi F) \oplus (\Pi M_2(F))$ as ordered pairs (u, v) with $u \in \Pi F$ and $v \in \Pi M_2(F)$. Let Λ be the subring of T generated by $(\oplus F) \oplus (\oplus M_2(F))$ and all elements of the form $(\text{long } a, \text{long} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix})$. The ring Λ is a left hereditary semiprime PI ring which is not even right p.p.

REFERENCES

1. G. Bergman, *Hereditary commutative rings and centres of hereditary rings*, Proc. London Math. Soc. (3) **23** (1971), 214–236.
2. J. Fuelberth, E. Kirkman and J. Kuzmanovich, *Hereditary module-finite algebras*, J. London Math. Soc. (2) **19** (1979), 268–276.
3. R. Gordon and L. W. Small, *Piecewise domains*, J. Algebra **23** (1972), 553–564.
4. S. Jøndrup, *The centre of a right hereditary ring*, J. London Math. Soc. (2) **15** (1977), 211–212.
5. ———, *Rings of quotients of some semiprime PI rings*, Comm. Algebra **4** (1976), 51–75.
6. ———, *Homological dimension of some P.I. rings*, Preprint Series, no. 16, København Universitet, København, 1979.
7. E. Kirkman and J. Kuzmanovich, *Orders over hereditary rings*, J. Algebra **55** (1978), 1–27.
8. R. S. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc., no. 70 (1967).
9. C. Procesi, *Rings with polynomial identities*, Dekker, New York, 1973.
10. J. C. Robson and L. W. Small, *Hereditary prime P.I. rings are classical hereditary orders*, J. London Math. Soc. (2) **8** (1974), 499–503. MR **50** #2236.
11. L. W. Small, *Semihereditary rings*, Bull. Amer. Math. Soc. **73** (1967), 656–658. MR **35** #2926.

DEPARTMENT OF MATHEMATICS, WAKE FOREST UNIVERSITY, WINSTON-SALEM, NORTH CAROLINA 27109