COMPACT WEIGHTED ENDOMORPHISMS OF $C(X)$

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ABSTRACT. A weighted endomorphism of an algebra is an endomorphism followed by a multiplier. In this note we characterize compact weighted endomorphisms of the Banach algebra $C(X)$, and also determine their spectra.

A *weighted endomorphism* of an algebra is defined to be a linear operator which is equal to an endomorphism followed by a multiplier. Thus, if B is an algebra with unit, then T is a weighted endomorphism of B if there is an element u in B and an endomorphism S of B such that $T: f \rightarrow u \cdot Sf$. The problem considered in this note is to characterize compact weighted endomorphisms of $C(X)$, the Banach algebra of continuous functions on a compact Hausdorff space X .

If X is a compact Hausdorff space, then it is easy to see that every nonzero endomorphism of $C(X)$ has the form $Sf = f \circ \varphi$ for some continuous function φ from X into X . Consequently, weighted endomorphisms of $C(X)$ have the form $f(x) \rightarrow u(x)f(\varphi(x))$ for some $u \in C(X)$ and continuous function φ from X into X . This map will be denoted by uC_φ and its spectrum by $\sigma(uC_\varphi)$.

In related papers, the spectra of weighted *automorphisms* of various algebras were considered by Kitover ([6] and [7]) and in a lattice theoretic setting by Arendt ([1] and [2]) and Schaeffer, Wolff and Arendt [8]. Results on weighted automorphisms and endomorphisms of the disc algebra may be found in [4] and [5].

We begin with some notation and terminology. If B_1 and B_2 are Banach spaces, then a linear map $T: B_1 \rightarrow B_2$ is *compact* if it maps bounded sets into sequentially compact sets. If X is a set and $\varphi: X \rightarrow X$, then we let φ_n denote the n th iterate of φ , i.e. $\varphi_0(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$ for $n > 0$, $x \in X$. Also, if $\varphi: X \rightarrow X$, then a point c in X is called a *fixed point of φ of order n* if n is a positive integer, $\varphi_n(c) = c$ and $\varphi_k(c) \neq c$, $k = 1, \dots, n-1$.

The principal result of this note will be the following theorem.

THEOREM A. *Suppose X is a compact Hausdorff space, $u \in C(X)$ and φ is a continuous function from X into X .*

(1) *The map $uC_\varphi: f(x) \rightarrow u(x)f(\varphi(x))$ is compact if, and only if, for each connected component C of $\{x \mid u(x) \neq 0\}$ there exists an open set $V \supset C$ such that φ is constant on V .*

(2) *If uC_φ is compact, then $\sigma(uC_\varphi) \setminus \{0\} = \{\lambda \mid \lambda^n = u(c) \dots u(\varphi_{n-1}(c)) \text{ for some positive integer } n \text{ and some fixed point } c \text{ of } \varphi \text{ of order } n\}$.*

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A key aid in the proof will be the following result which was kindly pointed out to me by M. Ling and T. Gamelin (private communication). The reader is referred to [3, Theorem VI, 7.1] for a full discussion.

THEOREM. *Suppose X is a compact Hausdorff space and T is a bounded linear map from a Banach space B into $C(X)$. Then there exists a function $\tau: X \rightarrow B^*$ satisfying $Tf(x) = \tau(x)f$ for all $f \in B$. In general, τ is continuous with the w^* -topology on B^* . A necessary and sufficient condition that T be a compact operator is that τ be continuous with the norm topology on B^* .*

If X is a compact Hausdorff space and $s \in X$, we will let e_s denote the element in $C(X)^*$ satisfying $e_s(f) = f(s)$ for all $f \in C(X)$. It is easy to show that if $T = uC_\varphi: f \rightarrow u \cdot f \circ \varphi$ is a weighted endomorphism of $C(X)$, then the corresponding map τ has the form $\tau(x) = u(x)e_{\varphi(x)}$ for all $x \in X$. Consequently the operator uC_φ is compact if, and only if, $(\text{net}) x_n \rightarrow x^*$ in X implies that $\|u(x_n)e_{\varphi(x_n)} - u(x^*)e_{\varphi(x^*)}\| \rightarrow 0$ in $C(X)^*$.

PROPOSITION 1. *Suppose X is a compact Hausdorff space, $u \in C(X)$ and φ is a continuous function from X into X . Assume uC_φ is a compact weighted endomorphism of $C(X)$. Then the following hold.*

- (1) *If $x^* \in X$ and $u(x^*) \neq 0$, then there exists an open set containing x^* on which φ is constant.*
- (2) *If $s \in X$ and $W = \{x \mid \varphi(x) \neq s\}$, then W has no boundary point in $\{x \mid u(x) \neq 0\}$.*
- (3) *If C is a connected component of $\{x \mid u(x) \neq 0\}$, then there exists an open set containing C on which φ is constant.*

PROOF. (1) Suppose $x^* \in X$, $u(x^*) \neq 0$ and φ is constant on no open set containing x^* . Then there is a net $\{x_n\}$ converging to x^* with the property that $\varphi(x_n) \neq \varphi(x^*)$. For each n , let $f_n \in C(X)$ with $\|f_n\| = 1$, $f_n(\varphi(x^*)) = 1$ and $f_n(\varphi(x_n)) = 0$. Then

$$\begin{aligned} \|\tau(x_n) - \tau(x^*)\| &\geq |\tau(x_n)f_n - \tau(x^*)f_n| \\ &= |u(x_n)f_n(\varphi(x_n)) - u(x^*)f_n(\varphi(x^*))| = |u(x^*)|. \end{aligned}$$

Since uC_φ is compact, it follows that τ is continuous with the norm topology on $C(X)^*$ and therefore $\|\tau(x_n) - \tau(x^*)\| \rightarrow 0$. This implies $u(x^*) = 0$, a contradiction. Hence if $u(x^*) \neq 0$, there exists an open set containing x^* on which φ is constant.

(2) Next let $s \in X$, $W = \{x \mid \varphi(x) \neq s\}$ and let x^* be a boundary point of W . Since W is open, $x^* \notin W$ and so $\varphi(x^*) = s$. If, in addition, $u(x^*) \neq 0$, then (1) implies that $\varphi(x) = s$ on some open set V containing x^* . But $V \cap W \neq \emptyset$ since x^* is a boundary point of W . This gives a contradiction, and hence $(\text{bdry } W) \cap \{x \mid u(x) \neq 0\} = \emptyset$.

(3) Let C be a connected component of $\{x \mid u(x) \neq 0\}$. We first show that φ is constant on C . Assume the contrary and let $x_1, x_2 \in C$ with $\varphi(x_1) = s_1 \neq s_2 = \varphi(x_2)$ and $W = \{x \mid \varphi(x) \neq s_1\}$. Then $W \cap C \neq \emptyset$ since $x_2 \in W \cap C$, $(\text{bdry } W) \cap C = \emptyset$ by (2), and $(X \setminus W) \cap C \neq \emptyset$ since $x_1 \in (X \setminus W) \cap C$. Therefore $\{(X \setminus \overline{W}) \cap C, W \cap C\}$ is a partition of C into two (relatively) open, disjoint,

nonempty subsets, contradicting the connectedness of C . Therefore φ is constant on C . Let $\varphi(x) = s$ for all $x \in C$. Since $C \subset \{x \mid u(x) \neq 0\}$, (1) implies that to each $x \in C$ there is an open set V_x , say, such that $x \in V_x$ and φ is constant on V_x , i.e. $\varphi(t) = s$ for $t \in V_x$. Thus if $V = \bigcup_{x \in C} V_x$, then V is open, $C \subset V$ and $\varphi(x) = s$ for all $x \in V$, as needed.

PROPOSITION 2. *Suppose X is a compact Hausdorff space, $u \in C(X)$ and φ is a continuous function from X to X . Suppose further that for each connected component C of $\{x \mid u(x) \neq 0\}$ there exists an open set containing C on which φ is constant. Then uC_φ is a compact operator.*

PROOF. We show that if the hypothesis is satisfied, then the map $\tau: x \rightarrow u(x)e_{\varphi(x)}$ is continuous with the norm topology on $C(X)^*$. To this end, let $x^* \in X$ and let $\{x_n\}$ be a net in X converging to x^* .

(i) Assume $u(x^*) = 0$. Since u is continuous, $u(x_n) \rightarrow u(x^*) = 0$. Now $\|u(x_n)\| = \|u(x_n)e_{\varphi(x_n)}\| = \|u(x_n)e_{\varphi(x_n)} - u(x^*)e_{\varphi(x^*)}\| = \|\tau(x_n) - \tau(x^*)\|$. Hence $\|\tau(x_n) - \tau(x^*)\| \rightarrow 0$, and so τ is continuous at x^* with the norm topology on $C(X)^*$ whenever $u(x^*) = 0$.

(ii) Assume $u(x^*) \neq 0$. By hypothesis there exists an open set containing x^* on which φ is a constant c , say. Then for x_n near x^* , $\|\tau(x_n) - \tau(x^*)\| = \|u(x_n)e_{\varphi(x_n)} - u(x^*)e_{\varphi(x^*)}\| = \|u(x_n)e_c - u(x^*)e_c\| = \|u(x_n) - u(x^*)\|$. Since u is continuous at x^* , we have that $\|u(x_n) - u(x^*)\| \rightarrow 0$. This shows that τ is continuous with the norm topology on $C(X)^*$ at each x^* with $u(x^*) \neq 0$.

Hence τ is continuous on X with the norm topology on $C(X)^*$ and therefore uC_φ is compact, as claimed.

Thus part (1) of Theorem A is proved.

Before turning to the question of the spectra of these maps we remark that for uC_φ to be compact on $C(X)$ it is not sufficient that φ be constant on each connected component of $\{x \mid u(x) \neq 0\}$. As an example, consider the subset X of the plane given by $X = \bigcup_n \{(1/n, y) \mid 0 < y < 1\} \cup \{(0, y) \mid 0 < y < 1\} \cup \{(x, 0) \mid 0 < x < 1\}$. On X let $u(x, y) = y$ and $\varphi(x, y) = (x, 0)$. Clearly $u(x, y) = 0$ if, and only if, $y = 0$. Since the connected components of $X_0 \equiv \{(x, y) \mid u(x, y) \neq 0\}$ are $\{(0, y) \mid 0 < y < 1\}$ and $\{(1/n, y) \mid 0 < y < 1\}$, all n , we have that φ is constant on each connected component of X_0 . Yet uC_φ is not a compact operator. Indeed, let $f_n(x, y) = 0$ if $1/n < x < 1$, $f_n(x, y) = 1 - nx$, $0 < x < 1/n$, $0 < y < 1$. Clearly $f_n \in C(X)$ and $\|f_n\| = 1$. Then $u(x, y)f_n(\varphi(x, y)) = u(x, y)f_n(x, 0) = y(1 - nx)$ if $0 < x < 1/n$, and $u(x, y)f_n(\varphi(x, y)) = 0$ if $1/n < x < 1$, all y . Now assume uC_φ is compact. Then there exist a continuous function G and a subsequence $\{f_{n_k}\}$ such that $u(x, y)f_{n_k}(\varphi(x, y)) \rightarrow G(x, y)$ uniformly. Since $u(0, 1)f_n(\varphi(0, 1)) = 1$ for all n , $G(0, 1) = 1$. On the other hand if $x \neq 0$, then $u(x, 1)f_{n_k}(\varphi(x, 1)) = f_{n_k}(x, 0) = 0$ for $1/n_k < x$. Hence $G(x, 1) = 0$ for $x \neq 0$. Thus G is not continuous at $(0, 1)$ and so uC_φ is not a compact operator.

PROPOSITION 3. *Suppose X is a compact Hausdorff space, $u \in C(X)$ and φ is a continuous function from X into X . Let n be a positive integer and c a fixed point of φ of order n . Then every complex number λ satisfying $\lambda^n = u(c) \dots u(\varphi_{n-1}(c))$ is in $\sigma(uC_\varphi)$.*

PROOF. If $u(\varphi_k(c)) = 0$ for some k , then uC_φ is not onto and so $\lambda = 0 \in \sigma(uC_\varphi)$.

Assume $u(\varphi_k(c)) \neq 0$ for all k and $\lambda^n = u(c) \dots u(\varphi_{n-1}(c))$.

If $n = 1$, then $\lambda = u(c)$ and $\varphi(c) = c$. Let $g \in C(X)$ satisfy $g(c) = 1$. Then no $f \in C(X)$ satisfies $\lambda f(x) - u(x)f(\varphi(x)) = g(x)$ for all x since $\lambda f(c) - u(c)f(\varphi(c)) = 0$ while $g(c) = 1$. Hence if $\varphi(c) = c$ and $\lambda = u(c)$, then $\lambda \in \sigma(uC_\varphi)$.

If $n \geq 2$, let $g \in C(X)$ satisfy

$$(*) \quad \lambda^{n-1}g(c) + \sum_{k=1}^{n-1} \lambda^{n-k-1}u(c) \dots u(\varphi_{k-1}(c))g(\varphi_k(c)) = 1.$$

Such a g can be found since none of the coefficients of $g(\varphi_j(c))$ in $(*)$ vanish. However, if $f \in C(X)$ satisfies $\lambda f(x) - u(x)f(\varphi(x)) = g(x)$, then by induction

$$(**) \quad \begin{aligned} & \lambda^n f(x) - u(x) \dots u(\varphi_{n-1}(x))f(\varphi_n(x)) \\ &= \lambda^{n-1}g(x) + \sum_{k=1}^{n-1} \lambda^{n-k-1}u(x) \dots u(\varphi_{k-1}(x))g(\varphi_k(x)), \end{aligned}$$

for all $x \in X$. Since $\varphi_n(c) = c$, evaluating $(**)$ at c gives

$$\begin{aligned} & [\lambda^n - u(c) \dots u(\varphi_{n-1}(c))]f(c) \\ &= \lambda^{n-1}g(c) + \sum_{k=1}^{n-1} \lambda^{n-k-1}u(c) \dots u(\varphi_{k-1}(c))g(\varphi_k(c)). \end{aligned}$$

In this equation, the left side equals 0 since $\lambda^n = u(c) \dots u(\varphi_{n-1}(c))$, while the right side equals 1. This contradiction shows that if λ satisfies $\lambda^n = u(c) \dots u(\varphi_{n-1}(c))$ for some fixed point c of order n , then λ is in $\sigma(uC_\varphi)$.

We remark that a similar result to Proposition 3 holds for weighted endomorphisms of any commutative semisimple Banach algebra. Also, if uC_φ is a compact weighted endomorphism, then each nonzero λ in $\sigma(uC_\varphi)$ in the preceding proposition is an eigenvalue.

In the remainder of this note we will show that if uC_φ is compact, then $\sigma(uC_\varphi) \setminus \{0\}$ consists only of the nonzero λ 's in Proposition 3.

PROPOSITION 4. Suppose X is a compact Hausdorff space, $u \in C(X)$, φ is a continuous function from X into X and uC_φ is a compact weighted endomorphism of $C(X)$. If $\lambda \neq 0$ and for all positive integers n and all fixed points c of φ of order n , $\lambda^n \neq u(c) \dots u(\varphi_{n-1}(c))$, then λ is not an eigenvalue of uC_φ .

PROOF. Fix $\lambda \neq 0$ such that, for all positive integers n , $\lambda^n \neq u(c) \dots u(\varphi_{n-1}(c))$ if c is a fixed point of φ of order n . Suppose $\lambda f = u \cdot f \circ \varphi$. We show that $f = 0$.

First suppose that c is a fixed point of φ of order n . We claim that $f(c) = 0$. Indeed, since $\lambda f = u \cdot f \circ \varphi$, we have $\lambda^n f(x) = u(x) \dots u(\varphi_{n-1}(x))f(\varphi_n(x))$ for all $x \in X$. Evaluating at c , we get $\lambda^n f(c) = u(c) \dots u(\varphi_{n-1}(c))f(\varphi_n(c))$. Since $\varphi_n(c) = c$ and $\lambda^n \neq u(c) \dots u(\varphi_{n-1}(c))$, we conclude that $f(c) = 0$.

Next let $b \in X$ and $\mathfrak{B} = \{b, \varphi(b), \varphi_2(b), \dots\}$.

(i) If $\mathfrak{B}$ is finite, then for some positive integers n and m , $m < n$, $\varphi_n(b) = \varphi_m(b)$. This implies $\varphi_{n-m}(\varphi_m(b)) = \varphi_m(b)$, or that $\varphi_m(b)$ is a fixed point of φ_{n-m} . By the preceding paragraph, $f(\varphi_m(b)) = 0$. Further, $\lambda^m f(x) = u(x) \dots u(\varphi_{m-1}(x))f(\varphi_m(x))$, and evaluating at b gives $f(b) = 0$.

(ii) If $\mathfrak{B}$ is *infinite*, then $\{\varphi_k(b)\}$ are distinct. Let $\delta > 0$. We claim first that $\{k \mid |u(\varphi_k(b))| > \delta\}$ is finite. For, if there exist infinitely many k satisfying $|u(\varphi_k(b))| > \delta$, then there is a cluster point a^* of $\{\varphi_k(b) \mid |u(\varphi_k(b))| > \delta\}$. Since u is continuous, $|u(a^*)| > \delta$. This implies that there exist infinitely many of these $\varphi_k(b)$ in V_{a^*} , an open set containing a^* on which φ is constant. If $\varphi_{k_1}(b)$ and $\varphi_{k_2}(b)$ are in V_{a^*} , then $\varphi(\varphi_{k_1}(b)) = \varphi(\varphi_{k_2}(b))$, contradicting the assumption that $\mathfrak{B}$ is infinite. Thus $\{k \mid |u(\varphi_k(b))| > \delta\}$ is finite.

Now choose $\delta > 0$ and K_1 with $\delta < |\lambda|$ and $|u(\varphi_k(b))| < \delta$ for $k \geq K_1$. Since $\lambda^N f(x) = u(x) \dots u(\varphi_{N-1}(x))f(\varphi_N(x))$, we have for $N > K_1$

$$|\lambda^N f(b)| = |u(b) \dots u(\varphi_{K_1-1}(b))u(\varphi_{K_1}(b)) \dots u(\varphi_{N-1}(b))| |f(\varphi_N(b))| \\ \leq \|u\|^K \delta^{N-K_1-1} |f(\varphi_N(b))|.$$

Thus $|f(b)| \leq \|u\|^{K_1} (\delta/|\lambda|)^N \delta^{-(K_1+1)} |f(\varphi_N(b))|$. Since $\delta < |\lambda|$, letting $N \rightarrow \infty$ gives $f(b) = 0$.

Thus if $\lambda \neq 0$ and $\lambda^n \neq u(c) \dots u(\varphi_{n-1}(c))$ for all n and all fixed points c of φ of order n , then $\lambda f = u \cdot f \circ \varphi$ implies that $f(x) = 0$ for all $x \in X$. That is, no such λ is an eigenvalue.

PROPOSITION 5. *Suppose X is a compact Hausdorff space, $u \in C(X)$ and φ is a continuous function from X into X . If uC_φ is compact, then $\sigma(uC_\varphi) \setminus \{0\} = \{\lambda \mid \lambda^n = u(c) \dots u(\varphi_{n-1}(c)) \text{ for some fixed point } c \text{ of } \varphi \text{ of order } n\}$.*

The proof of this follows from Propositions 3 and 4 since every nonzero element in the spectrum of a compact operator is an eigenvalue.

Finally, Theorem A is just a restatement of Propositions 1, 2 and 5.

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