HARDY'S INEQUALITIES WITH INDICES LESS THAN 1

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ABSTRACT. In this paper Hardy inequalities are proved between weighted L^p , L^q spaces with indices p, q less than 1. The results are almost characterizations of those weights for which weighted estimates hold.

In this note we prove weighted Hardy inequalities with mixed indices less than 1. The results are almost characterizations of those weights for which weighted estimates hold. Our theorems correspond closely to the characterizations given by Bradley [2] (and independently by Andersen and Muckenhoupt) for indices p, q with $1 < p \leq q \leq \infty$, which are extensions of earlier results of Muckenhoupt [5] for the case $p = q > 1$. Furthermore, they complement and extend some of those recently given in [4].

Let u, v and f denote nonnegative extended real valued measurable functions on $(0, \infty)$. For $r > 0$ and $p, q < 1, p \neq 0, q \neq 0$, we define K and J by

$$K(r) = \left(\int_0^r u(x)^q dx \right)^{1/q} \left(\int_0^r v(x)^{-p'} dx \right)^{1/p'}$$

and

$$J(r) = \left(\int_r^\infty u(x)^q dx \right)^{1/q} \left(\int_r^\infty v(x)^{-p'} dx \right)^{1/p'},$$

where here and throughout the indices p and p' are related by $1/p + 1/p' = 1$. Moreover, we adhere to the convention that products of the form $0(+\infty)$ are taken to be zero.

In the sequel we consider the (Hardy) operator P and its dual Q , defined by $(Pf)(x) = \int_0^x f(t) dt$ and $(Qf)(x) = \int_x^\infty f(t) dt, 0 < x < \infty$.

A function f is said to be P -admissible, respectively Q -admissible, if $(Pf)(x)$, respectively $(Qf)(x)$, is finite for $0 < x < \infty$.

In the next two theorems we give analogues for $p, q < 1, p \neq 0, q \neq 0$ of results of [2].

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THEOREM 1. (a) Let $p, q < 0$ and $0 < \int_0^r v(t)^{-p'} dt < \infty$ for all $r > 0$. If there exists a $C > 0$ such that

$$(1) \quad \left(\int_0^\infty [v(x)f(x)]^p dx \right)^{1/p} \leq C \left(\int_0^\infty [u(x)(Pf)(x)]^q dx \right)^{1/q}$$

holds for all admissible f , then

$$(2) \quad \inf_{r>0} K(r) \equiv B > 0.$$

If $q < p < 0$ and (2) holds with $K(r)$ nondecreasing, then there exists a $C > 0$ such that (1) holds for all admissible f .

(b) Let $p, q < 0$, and $0 < \int_r^\infty v(t)^{-p'} dt < \infty$ for all $r > 0$. If there exists a $C > 0$ such that for all admissible f we have

$$(3) \quad \left(\int_0^\infty [v(x)f(x)]^p dx \right)^{1/p} \leq C \left(\int_0^\infty [u(x)(Qf)(x)]^q dx \right)^{1/q},$$

then

$$(4) \quad \inf_{r>0} J(r) \equiv B > 0.$$

If $q < p < 0$ and (4) holds with nonincreasing $J(r)$, then (3) holds for some C and all admissible f .

Moreover, the smallest C in (1) and (3) satisfies $B \geq C^{-1} \geq |p|^{1/q}(p')^{1/p'}B$ whenever $q < p < 0$.

PROOF. (a) From (1) we obtain

$$(5) \quad \int_0^\infty u(x)^q \left(\int_0^x f(t) dt \right)^q dx \leq C^{-q} \left(\int_0^\infty [v(x)f(x)]^p dx \right)^{q/p}.$$

Fix $r > 0$, $k > 0$ and let $f(x) = v(x)^{-p'}$ on $(0, r]$ and $f(x) = kf_1(x)$ on (r, ∞) , where f_1 is any P -admissible function for which $\int_r^\infty [v(x)f_1(x)]^p dx < \infty$. Then $\int_0^x f(t) dt = \int_0^x v(t)^{-p'} dt$ if $0 < x \leq r$ and it is equal to $\int_0^r v(t)^{-p'} dt + k \int_r^x f_1(t) dt$ if $x > r$, and from (5) we obtain

$$\int_0^r u(x)^q \left(\int_0^x v(t)^{-p'} dt \right)^q dx \leq C^{-q} \left(\int_0^r v(t)^{-p'} dt + k^p \int_r^\infty [v(t)f_1(t)]^p dt \right)^{q/p}.$$

Since $q < 0$, $p < 0$, then as $k \rightarrow +\infty$ we obtain for any $r > 0$

$$\left(\int_0^r u(x)^q dx \right) \left(\int_0^r v(t)^{-p'} dt \right)^q \leq C^{-q} \left(\int_0^r v(t)^{-p'} dt \right)^{q/p},$$

which implies (2) with $B \geq C^{-1}$.

For the (near) converse, let $h(t)^{pp'} = \int_0^t v(s)^{-p'} ds$. Then by Hölder's inequality for any admissible f

$$\begin{aligned} \int_0^x f(t) dt &\geq \left(\int_0^x [v(t)f(t)h(t)]^p dt \right)^{1/p} \left(\int_0^x [v(t)h(t)]^{-p'} dt \right)^{1/p'} \\ &= \left(\int_0^x [v(t)f(t)h(t)]^p dt \right)^{1/p} \left(\int_0^x v(t)^{-p'} \left(\int_0^t v(s)^{-p'} ds \right)^{-1/p} dt \right)^{1/p'} \\ &= (p')^{1/p'} h(x)^{p/p'} \left(\int_0^x [v(t)f(t)h(t)]^p dt \right)^{1/p}, \end{aligned}$$

where the last equality is obtained on integrating. Hence if $q < 0$,

$$\begin{aligned} I &\equiv \int_0^\infty [u(x)(Pf)(x)]^q dx \\ &< (p')^{q/p'} \int_0^\infty u(x)^q h(x)^{pq/p'} \left(\int_0^x [v(t)f(t)h(t)]^p dt \right)^{q/p} dx \\ &= (p')^{q/p'} \int_0^\infty \left(\int_0^\infty \chi_{[0,x]}(t) [v(t)h(t)f(t)]^p dt \right)^{q/p} d\mu(x), \end{aligned}$$

where $d\mu(x) = u(x)^q h(x)^{pq/p'} dx$ and $\chi_{[0,x]}$ is the characteristic function over $[0, x]$. Now, by Minkowski's inequality with parameter $q/p > 1$,

$$\begin{aligned} I &\leq (p')^{q/p'} \left(\int_0^\infty \left[\int_0^\infty \chi_{[0,x]}(t) [v(t)h(t)f(t)]^q d\mu(x) \right]^{p/q} dt \right)^{q/p} \\ &= (p')^{q/p'} \left(\int_0^\infty [v(t)h(t)f(t)]^p \left(\int_t^\infty u(x)^q h(x)^{pq/p'} dx \right)^{p/q} dt \right)^{q/p}. \end{aligned}$$

But by the definition of K , $h(x)^{pq/p'} = ([\int_0^x v(s)^{-p'} ds]^{1/p'})^{q/p'} = K(x)^{q/p'} (\int_0^x u(s)^q ds)^{-1/p'}$ so that

$$\begin{aligned} I &\leq (p')^{q/p'} \left(\int_0^\infty [v(t)h(t)f(t)]^p \right. \\ &\quad \cdot \left. \left[\int_t^\infty u(x)^q K(x)^{q/p'} \left(\int_0^x u(s)^q ds \right)^{-1/p'} dx \right]^{p/q} dt \right)^{q/p} \\ &\leq (p')^{q/p'} \left(\int_0^\infty [v(t)h(t)f(t)]^p K(t)^{p/p'} \right. \\ &\quad \cdot \left. \left[\int_t^\infty \left(\int_0^x u(s)^q ds \right)^{-1/p'} u(x)^q dx \right]^{p/q} dt \right)^{q/p}, \end{aligned}$$

since K is nondecreasing and $q/p' < 0$. On integrating, the inner integral is dominated by $-p(\int_0^t u(s)^q ds)^{1/p}$, which by definition of K is equal to $-pK(t)^{q/p}(\int_0^t v(s)^{-p'} ds)^{-q/(pp')} = -pK(t)^{q/p}h(t)^{-q}$. Hence by (2), since $p < 0$

$$\begin{aligned} I &\leq (-p)(p')^{q/p'} \left(\int_0^\infty [v(t)h(t)f(t)]^p K(t)^{1+(p/p')} h(t)^{-p} dt \right)^{q/p} \\ &\leq (-p)(p')^{q/p'} B^q \left(\int_0^\infty [v(t)f(t)]^p dt \right)^{q/p}. \end{aligned}$$

Thus (1) holds with $C^{-1} > (-p)^{1/q}(p')^{1/p'}B$.

The proof of (b) is almost the same. In showing that (3) implies (4), set $f(x) = kf_1(x)$ on $(0, r]$ and $f(x) = v(x)^{-p'}$ for $x \in (r, \infty)$, $r > 0$, $k > 0$ and the argument proceeds as in part (a). For the sufficiency part, define h by $h(t)^{pp'} = \int_t^\infty v(s)^{-p'} ds$ and the argument again follows as in part (a).

We remark that for $q = p$ and $K(r) \equiv B$ the constant in (1) with $C^{-1} = |p|^{1/p}(p')^{1/p'}B$ is best possible. Equality is obtained in the application of Hölder's inequality in the proof if and only if $f(t) = f_0(t) \equiv k_1 v(t)^{-p'} (\int_0^t v(s)^{-p'} ds)^{-1/p}$,

where k_1 is some constant. A simple calculation then shows that for $0 < \alpha < \beta < \infty$

$$\left\{ \int_{\alpha}^{\beta} [v(x)f(x)]^p dx / \int_{\alpha}^{\beta} [u(x)Pf(x)]^p dx \right\}^{1/p} = (Bp')^{-1}(1-p)^{-1/p} = C.$$

To see that this C is best possible, one must for given $\delta > 0$ find an admissible function $f_1 > 0$, such that

$$\int_0^{\infty} [u(x)Pf_1(x)]^p dx > (C^{-p} - \delta) \int_0^{\infty} [v(x)f_1(x)]^p dx.$$

Let f_0 be defined as above; then following Beesack [1, pp. 52-54] one can show that the function f_1 defined by

$$f_1(x) = \begin{cases} v(x)^{-p'} \left(\int_0^{\alpha} u(t)^p dt \right)^{1/(p(p-1))}, & 0 \leq x < \alpha, \\ f_0(x), & \alpha \leq x \leq \beta, \\ k(x), & \beta < x < \infty, \end{cases}$$

for suitable α and β and any fixed admissible function k , satisfies this estimate. We omit the details. Observe that equality never holds in (1) in this case.

In the next theorem, we restrict the indices to the interval $(0, 1)$.

THEOREM 2. (a) Suppose $0 < p, q < 1$ and for all $r > 0$, $0 < \int_r^{\infty} u(x)^q dx < \infty$, $0 < \int_r^{\infty} v(x)^{-p'} dx < \infty$. If (1) holds for all admissible f , then (4) holds and $B > C^{-1}$.

Let $0 < q \leq p < 1$. If $J(r)$ is nonincreasing and satisfies (4), then (1) holds for all admissible f with $C^{-1} > p^{1/q} |p'|^{1/p'} B$.

(b) Let $0 < p, q < 1$ and for all $r > 0$, $0 < \int_r^{\infty} u(x)^q dx < \infty$, $0 < \int_0^r v(x)^{-p'} dx < \infty$. If (3) holds for some $C > 0$ and all admissible f , then (2) holds with $B > C^{-1}$. Moreover, if $0 < q \leq p < 1$ and $K(r)$ is nondecreasing and satisfies (2), then (3) holds for all admissible f and $C^{-1} > p^{1/q} |p'|^{1/p'} B$.

PROOF. The proof is quite similar to that of Theorem 1. (See also [4, Theorem 4a]). We therefore sketch only the proof of the first part of part (a).

Rewrite (1) so that

$$(6) \quad \int_0^{\infty} [v(x)f(x)]^p dx \leq C^p \left(\int_0^{\infty} [u(x)(Pf)(x)]^q dx \right)^{p/q}.$$

Now let $r > 0$ be fixed and define $f(x) = v(x)^{-p'}$ for $x \in [r, \infty)$ and zero otherwise. Then from (6)

$$\begin{aligned} \int_r^{\infty} v(x)^{-p'} dx &= \int_r^{\infty} [v(x)v(x)^{-p'}]^p dx \\ &\leq C^p \left(\int_r^{\infty} u(x)^q \left(\int_r^x v(t)^{-p'} dt \right)^q dx \right)^{p/q} \\ &\leq C^p \left(\int_r^{\infty} u(x)^q dx \right)^{p/q} \left(\int_r^{\infty} v(t)^{-p'} dt \right)^p, \end{aligned}$$

so that $C^{-1} \leq (\int_r^{\infty} u(x)^q dx)^{1/q} (\int_r^{\infty} v(t)^{-p'} dt)^{1/p'}$. Hence (4) holds with $C^{-1} \leq B$.

Theorems 1, 2 with $p = q$ and $(u, v) = (x^{-k/p}, x^{1-(k/p)})$ and $C^{-1} = |p|^{1/q}(p')^{1/p'}B = |p/(k-1)|$, yield the cases $p < 0$, $0 < p < 1$ of Hardy's inequalities [1, pp. 54, 57] and (for $0 < p < 1$) [3, Theorem 347], and these inequalities are best possible.

We conclude with a special case $0 < p = q < 1$. For this we give an inequality of the form (1) with v as an explicit expression in terms of u .

COROLLARY. Suppose $0 < p < 1$ and $\int_0^\infty u(x)(Pf)(x)^p dx < \infty$; then $\int_0^\infty v(x)f(x)^p dx < \infty$, where $v(x) = u(x)^{1-p}(\int_x^\infty u(t) dt)^p$.

Moreover

$$(7) \quad \int_0^\infty v(x)f(x)^p dx \leq p^{-p} \int_0^\infty u(x)(Pf)(x)^p dx$$

and the constant is best possible.

By replacing u, v here by u^p and v^p , (7) follows from Theorem 2(a), since an easy computation in this case shows that $J(r) = (1-p)^{1/p'} = B$ is constant. Thus (1) holds with $(q = p) C^{-1} = p^{1/p}|p'|^{1/p'}B = p$, so that (7) follows.

The following alternative proof of (7) may be of interest since it is easier: An integration, an interchange of order of integration and Hölder's inequality show that

$$\begin{aligned} p^{-1} \int_0^\infty u(x) \left(\int_0^x f(t) dt \right)^p dx &= \int_0^\infty u(x) \int_0^x \left(\int_0^t f(s) ds \right)^{p-1} f(t) dt dx \\ &= \int_0^\infty f(t) u(t)^{-1/p' + 1/p'} \left(\int_0^t f(s) ds \right)^{p-1} \left(\int_t^\infty u(x) dx \right) dt \\ &> \left(\int_0^\infty v(t) f(t)^p dt \right)^{1/p} \left(\int_0^\infty u(t) \left(\int_0^t f(s) ds \right)^p dt \right)^{1/p'}. \end{aligned}$$

Hence

$$\left(\int_0^\infty v(t) f(t)^p dt \right)^{1/p} \leq p^{-1} \left(\int_0^\infty u(x) \left(\int_0^x f(t) dt \right)^p dx \right)^{1-1/p'}$$

and the result follows.

A similar result with P replaced by Q holds; however in this case $v(x) = u(x)^{1-p}(\int_0^x u(t) dt)^p$.

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