A DECOMPOSITION THEOREM

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ABSTRACT. We prove a generalization of the following well-known theorem: " $\mathbf{R}$ is the disjoint union of a set of first category and a set of Lebesgue measure zero."

The following theorem is well known [5, p. 4].

THEOREM 1. *$\mathbf{R}$ is the disjoint union of a set of first category and a set of Lebesgue measure zero.*

This result remains true in some measure spaces, which have a base, whose cardinal has measure zero [5, p. 64]. By means of the abstract Baire category theory, which was developed in [2–4], we present in this note a new, quite different generalization of Theorem 1. To prove our result we suppose that the reader is familiar with the concept of $\mathfrak{B}$ - and $\mathfrak{C}$ -families, introduced in [3 and 2], respectively.

In the sequel let X be a topological group, which is a complete, separable, metric space without isolated points.

THEOREM 2. *If $\mathcal{C}$ and $\mathcal{D}$ are nonequivalent $\mathfrak{B}$ - and $\mathfrak{C}$ -families on X (simultaneously), then X is the disjoint union of a $\mathcal{C}_I$ -set and a $\mathcal{D}_I$ -set.*

PROOF. Since $\mathcal{C}$ and $\mathcal{D}$ are nonequivalent, there are two possibilities (see the remark after Definition 5 in [4]):

- (1) there is a $\mathcal{D}$ -set A , which contains no $\mathcal{C}$ -set, or
- (2) there is a $\mathcal{C}$ -set C , which contains no $\mathcal{D}$ -set.

In the first case suppose that A is not $\mathcal{C}$ -singular. But then by Theorem 8 in [3] the perfect set A contains a $\mathcal{C}$ -set, which is not possible. Hence A is $\mathcal{C}$ -singular. Now let D be a countable, topologically dense subset of X . Then

$$B := A + D = \bigcup \{A + t : t \in D\} \in \mathcal{C}_I.$$

By Theorem 4 in [4] and by Theorem 6 in [3] we have $A \in \mathcal{D} \subset \mathfrak{B}(\mathcal{D}) \cap \mathcal{D}_{II}$. Thus the corollary in [2, p. 64] yields that the complement of B is a $\mathcal{D}_I$ -set. (We here remark that the mentioned corollary is not only valid in $X = \mathbf{R}$, but also for every topological group X with the property that each topologically dense subset of X contains a countable, topologically dense subset.)

The second case can be treated in the same manner.

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Theorem 1 can be obtained from Theorem 2 by setting

$\mathcal{C} := \{ \{x \in \mathbf{R} : |x - y| \leq \frac{1}{n} \} : y \in \mathbf{R}, n \in \mathbf{N} \}$ and

$\mathcal{D} := \{ A \subset \mathbf{R} : A \text{ is closed, } \forall x \in A \forall U \in \tau x : \mu(U \cap A) > 0 \}$

(τx is the neighbourhood filter of x and μ denotes the Borel measure in $\mathbf{R}$). By Examples 4 (modified as above) and 5 in [3, p. 428], $\mathcal{C}$ and $\mathcal{D}$ are (translation invariant) $\mathfrak{B}$ -families, such that $\mathcal{C}_1$ is the family of sets of first category and $\mathcal{D}_1$ is the family of sets of Lebesgue measure zero. If $A, B \in \mathcal{C}$, respectively, $A, B \in \mathcal{D}$ and if D is a topologically dense subset of $\mathbf{R}$, then in the first case the Baire category theorem yields that $A \cap (B + D)$ is of second category. In the second case $U := \{x \in \mathbf{R} : \mu(A \cap (x + B)) > 0\}$ is a nonempty open set [1, p. 296] and thus $A \cap (B + D)$ is of positive Lebesgue measure. Therefore $\mathcal{C}$ and $\mathcal{D}$ are $\mathfrak{B}$ - and $\mathfrak{S}$ -families, which are nonequivalent, since, by example, any Cantor set of positive Lebesgue measure is nowhere dense.

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