EXISTENCE OF SETS OF UNIQUENESS OF l^p FOR GENERAL ORTHONORMAL SYSTEMS¹

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ABSTRACT. It is proved that for every orthonormal complete system in $L^2(0, 1)$ there exists a set A , of measure arbitrarily close to 1, which carries no nonzero function with Fourier transform in l^p , for every $p < 2$.

1. Suppose $\{\phi_n\}_{n=1}^\infty$ is an orthonormal complete system (ONC) in $L^2(0, 1)$. We call a Lebesgue measurable set $E \subset (0, 1)$ a set of uniqueness of l^p if no nonzero function $f \in L^2(0, 1)$, vanishing almost everywhere in the complement of E , satisfies the condition

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p < +\infty,$$

where $\{\hat{f}(n)\}_{n=1}^\infty$ denotes the Fourier transform of f with respect to the system $\{\phi_n\}$, i.e.

$$\hat{f}(n) = \int_0^1 f(x) \overline{\phi_n(x)} \, dx.$$

Y. Katznelson [6] first proved that the trigonometric system admits sets of uniqueness of l^p , for every $p < 2$, of Lebesgue measure arbitrarily close to 1 (see also [3]). Katznelson's theorem has been subsequently generalized to the system of characters of a nondiscrete locally compact abelian group by A. Figà-Talamanca and G. I. Gaudry [4], and to every uniformly bounded ONC by the author [2].

The aim of this paper is to prove a further extension of this result to every ONC. As a consequence we give a new proof of the generalization (due to W. Orlicz and A. M. Olevskiĭ) of a well-known theorem of Carleman stating that there exists a continuous function f such that

$$\sum_{n=1}^{\infty} |\hat{f}(n)|^p = +\infty \quad \text{for every } p < 2.$$

2. For $1 \leq p \leq +\infty$ we use $\|f\|_p$ and $\|\hat{f}\|_p$ in their usual meanings. The following lemmas hold.

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LEMMA 1. Suppose $\phi_1, \dots, \phi_N$ are functions in $L^2(0, 1)$, and E is an interval contained in $(0, 1)$. If $\varepsilon > 0$ and $\delta > 0$, there exists a function $\Psi \in L^2(0, 1)$ such that:

- (i) $\Psi(x) = 0$ if $x \notin E$;
- (ii) $|\{x \in E/\Psi(x) \neq 1\}| < \delta|E|$;
- (iii) $\|\Psi\|_2 < (2|E|/\delta)^{1/2}$;
- (iv) $|\int_0^1 \Psi(x)\phi_j(x) dx| < \varepsilon, j = 1, \dots, N$.

PROOF. Let $k = [1/\delta] + 1$ and let n be a positive integer. We split E in k^n intervals $E_1, \dots, E_{k^n}$ of the same measure. Set

$$\begin{aligned}\psi_n(x) &= 0 && \text{if } x \notin E, \\ &= 1 - k && \text{if } x \in E_1 \cup E_{k+1} \cup E_{2k+1} \cup \dots \cup E_{k^n-k+1}, \\ &= 1 && \text{if } x \in E \setminus E_1 \cup E_{k+1} \cup \dots \cup E_{k^n-k+1}.\end{aligned}$$

A direct computation shows that ψ_n 's satisfy (i)–(iii) for every n ; moreover, $\{\psi_n\}_{n=1}^\infty$ tends to 0 weakly, and putting $\Psi = \psi_n$, with n large enough, (iv) is satisfied too.

□

LEMMA 2. Suppose $\{\phi_n\}$ is an ONC. Then, for every $\delta > 0$, $q > 2$, $0 < a < 1$, there exists a function $\Psi \in L^2(0, 1)$ such that:

- (i) $\Psi(x) = 0$ if $x \notin (0, a)$;
- (ii) $|\{x \in (0, a)/\Psi(x) \neq 1\}| < \delta$;
- (iii) $\|\hat{\Psi}\|_q < \delta$.

PROOF. Let ε and η be positive numbers to be specified later. Divide $(0, a)$ into m intervals $E_1, \dots, E_m$ of measure less than η . We shall define the required function Ψ piecewise on every E_i .

Let

$$\begin{aligned}\psi_1(x) &= 1 && \text{if } x \in E_1, \\ &= 0 && \text{if } x \notin E_1,\end{aligned}$$

and put $n_1 = 1$.

Suppose now $\psi_1, \dots, \psi_{i-1}$ have already been defined. Then there exists an integer n_i such that

$$(1) \quad \left| \sum_{j=1}^{i-1} \hat{\psi}_j(n) \right| < \varepsilon \quad \text{for every } n \geq n_i,$$

and, via Lemma 1, it is possible to construct a function $\psi_i \in L^2(0, 1)$ such that:

$$\begin{aligned}\psi_i(x) &= 0 && \text{if } x \notin E_i; \\ |\{x \in E_i/\psi_i(x) \neq 1\}| &< \delta|E_i|; \\ \|\psi_i\|_2 &< (2|E_i|/\delta)^{1/2}, \text{ and}\end{aligned}$$

$$(2) \quad |\hat{\psi}_i(n)| < \varepsilon/2^i \quad \text{for every } n \leq n_i.$$

Put $\Psi = \sum_{i=1}^m \psi_i$. It is easy to see that Ψ satisfies (i) and (ii). Moreover,

$$(3) \quad \|\Psi\|_2 < (2/\delta)^{1/2}.$$

In order to prove (iii) we observe that for every i and every n ,

$$(4) \quad |\hat{\psi}_i(n)| \leq \|\psi_i\|_2 < (2|E_i|/\delta)^{1/2} < (2\eta/\delta)^{1/2} < \varepsilon$$

if $\eta = \eta(\varepsilon)$ is chosen small enough.

Then, if $n \geq n_m$, from (1) and (4) it follows that

$$|\hat{\Psi}(n)| \leq \left| \sum_{j=1}^{m-1} \hat{\psi}_j(n) \right| + |\hat{\psi}_m(n)| < 2\varepsilon,$$

and, if $n_{i-1} \leq n < n_i$,

$$|\hat{\Psi}(n)| \leq \left| \sum_{j=1}^{i-2} \hat{\psi}_j(n) \right| + |\hat{\psi}_{i-1}(n)| + \sum_{j=i}^m |\hat{\psi}_j(n)| = I_1 + I_2 + I_3.$$

But, it follows from (1) that $I_1 < \varepsilon$, from (4) that $I_2 < \varepsilon$, and from (2) that $I_3 < \sum_{j=i}^m (\varepsilon/2^j) < \varepsilon$. Collecting these results we obtain

$$(5) \quad \|\hat{\Psi}\|_\infty < 3\varepsilon,$$

and so, from (3) and (5),

$$\|\hat{\Psi}\|_q \leq \|\hat{\Psi}\|_2^{2/q} \cdot \|\hat{\Psi}\|_\infty^{(q-2)/q} < (2/\delta)^{1/q} \cdot (3\varepsilon)^{(q-2)/q} < \delta$$

if ε is small enough. $\square$

REMARK 1. This lemma was originally proved by Y. Katznelson for the trigonometric system, and subsequently extended to any uniformly bounded ONC by A. Figà-Talamanca and G. I. Gaudry. Our proof, which holds for any, possibly unbounded, ONC is based on an idea of G. Alexits (see [1, Chapter II, §11]).

THEOREM. For every ONC $\{\phi_n\}$, and every $\varepsilon > 0$, there exists a measurable set $A \subset (0, 1)$, with $|A| < \varepsilon$, such that if $f \in L^2(0, 1)$ vanishes a.e. in A , and $\|\hat{f}\|_p < +\infty$ for some $p < 2$, then $f(x) = 0$ a.e. in $(0, 1)$.

The theorem follows from Lemma 2 as in [2].

REMARK 2. I. I. Hirschman and Y. Katznelson [5] proved that the trigonometric system admits closed sets which are sets of uniqueness of l^p , but not of $l^{p'}$, with $p < p' < 2$. For an arbitrary ONC this feature fails to hold, as is shown in [2].

3. It is interesting to notice that, using our theorem, it is possible to prove easily the extension of a well-known theorem of T. Carleman to every ONC (see [7, Chapter III, §4] for the original proof of this extension).

THEOREM. For every ONC $\{\phi_n\}$ there exists a continuous bounded function f such that $\|\hat{f}\|_p = +\infty$ for every $p < 2$.

PROOF. See [2].

REMARK 3. The theorems stated for orthonormal systems in $L^2(0, 1)$ can be easily extended to orthonormal systems in $L^2(-\infty, +\infty)$ or to orthonormal systems of square integrable functions over more general measure spaces.

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