SOME QUOTIENTS OF THE STEENROD ALGEBRA

DONALD M. DAVIS¹

ABSTRACT. The quotient of the Steenrod algebra obtained by killing all except one of the generators Sq^{2^j} is studied.

The mod 2 Steenrod algebra A is generated by the elements Sq^{2^j} for $j \geq 0$. In this paper we consider the quotients obtained by killing all except one of these generators. Despite the large size of A these quotients have at most one nonzero element in each degree.

THEOREM 1. Let $N_i = A/A\{Sq^{2^j} : j \neq i\}$. Then

$$(\Sigma^{2^{i+1}-1}N_i)_k = \begin{cases} \mathbb{Z}_2 & \text{if } \alpha(k) = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\alpha(k)$ denotes the number of 1's in the binary expansion of k . Also, if M is an A -module, then M_k is its component in degree k and $(\Sigma^l M)_k = M_{k-l}$.

The nonzero elements of N_i admit a nice description. Let $a_{l,j} = Sq^{2^{j-1}} \cdots Sq^{2^{j+1}} Sq^{2^j}$ if $l > j$, and $a_{j,j} = 1$.

THEOREM 2. The nonzero elements of N_i are $\overline{a_{l_0,0} \cdots a_{l_i,i}}$ with $0 \leq l_0 < \cdots < l_i$.

For example, the nonzero elements of N_1 are all

$$Sq^{2^{l_0-1}} \cdots Sq^2 Sq^1 Sq^{2^{l_1-1}} \cdots Sq^4 Sq^2$$

with $0 \leq l_0 < l_1$.

Let χ denote the canonical antiautomorphism and $N'_i = A/A\{\chi Sq^{2^j} : j \neq i\}$.

THEOREM 3. $A\{\chi Sq^{2^j} : j \neq i\} = A\{Sq^{2^j} : j \neq i\}$, so that N_i and N'_i are quotients by the same submodule.

Let P_1 denote the A -module $\tilde{H}^*(RP^\infty)$, i.e. $(P_1)_k = \mathbb{Z}_2$ for all $k \geq 0$ with generator x^k and $Sq^n x^k = \binom{k}{n} x^{k+n}$. Let L_i denote the A -submodule generated by $\{x^j : j < 2^i\}$. The next result, from which Theorem 1 follows immediately, shows the way in which each N_i is faithfully represented in P_1 .

THEOREM 4. (i) There is an isomorphism of A -modules $\Sigma^{2^{i+1}-1}N_i \rightarrow L_{i+1}/L_i$.

(ii) $L_i = \langle x^k : \alpha(k) \leq i \rangle$.

In order to prove these theorems, it is useful to work in the dual A^* . Let $\langle t, e \rangle = \xi_{i+1-e}^{2^i}$, where $A^* = \mathbb{Z}_2[\xi_1, \dots]$ [3].

Received by the editors January 26, 1981.

1980 *Mathematics Subject Classification.* Primary 55S10.

Key words and phrases. Steenrod algebra.

¹This work was supported by National Science Foundation research grant MCS-7900293.

© 1981 American Mathematical Society
0002-9939/81/0000-0540/\$01.75

THEOREM 5. $\chi N_i^* = \langle S_i(e_0, \dots, e_k) : k \geq 0, 0 \leq e_0 < e_1 < \dots < e_k \leq i + k \rangle$, where $S_i(e_0, \dots, e_k) = \sum_{\sigma \in \mathcal{P}(k)} \prod_{l=0}^k \langle i + l, e_{\sigma(l)} \rangle$. Here $\mathcal{P}(k)$ is the group of permutations of $\{0, \dots, k\}$.

These results were noticed during the author's attempt to prove the Segal conjecture for $Z_2 \times Z_2$ [2]. There the analogous results for A_r were used, where A_r is the subalgebra of A generated by Sq^j with $j \leq 2^r$. All five results have analogues with the only changes being that: (i) $\alpha(k)$ is replaced by $\beta_r(k)$, defined by $\beta_r(a2^{r+1} + b) = a + \alpha(b)$ if $0 \leq b < 2^{r+1}$; (ii) in Theorem 2 we have $l_i \leq 2^{r+1}$, and $l_j < l_{j+1}$ unless $l_{j+1} = r + 1$, in which case $l_j < l_{j+1}$; and (iii) in Theorem 5, $k \leq r$, $i + k$ and $i + l$ are replaced by $\min(i + k, r)$ and $\min(i + l, r)$, and $\sigma \in \mathcal{P}'(k)$, the set of permutations of $\{0, \dots, k\}$ which are increasing on $\{r - i, \dots, k\}$ if $r - i < k$.

We begin by proving that χN_i^* is the vector space described in Theorem 5. Recall the formulas [1] $\xi_i Sq = \xi_i + \xi_{i-1}$ and $Sq \xi_i = \xi_i + \xi_{i-1}^2$, where $Sq = \sum_{j \geq 0} Sq^j$. These actions will also be denoted by $R Sq$ and $L Sq$, respectively. $N_i^* = \bigcap_{j \neq i} \ker(L Sq^{2^j} : A^* \rightarrow A^*)$ and hence

$$\chi N_i^* = \bigcap_{j \neq i} \ker(R Sq^{2^j} : A^* \rightarrow A^*).$$

Since

$$\langle t, e \rangle Sq^{2^j} = \begin{cases} \langle t - 1, e \rangle & \text{if } j = t, \\ 0 & \text{otherwise,} \end{cases}$$

it is easily shown that a polynomial in the $\langle t, e \rangle$'s is in $\bigcap_{j < i} \ker R Sq^{2^j}$ if and only if all t 's are $\geq i$.

If $j > i$, then $S_i(e_0, \dots, e_k) Sq^{2^j} = 0$. This is clear if $j > i + k$, while if $j \leq i + k$ it follows because all terms appear twice in $\sum_{\sigma \in \mathcal{P}(k)} \prod_{l=0}^k \langle d_j(i + l), e_{\sigma(l)} \rangle$, where

$$d_j(m) = \begin{cases} m - 1 & \text{if } m = j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $S_{r,i}(e_0, \dots, e_k) \in \chi N_i^*$.

Next we show that any element in χN_i^* must contain some $S_i(e_0, \dots, e_k)$. Since the proof below of Theorem 4(i) shows that all $S_i(e_0, \dots, e_k)$ have distinct degree, this implies that the element of χN_i^* equals $S_i(e_0, \dots, e_k)$.

Suppose P is a polynomial in the $\langle t, e \rangle$'s in $\bigcap_{j \neq i} \ker R Sq^{2^j}$. We noted above that all $t \geq i$. Suppose j is minimal such that P has a monomial M containing $\langle j, e_1 \rangle \langle j, e_2 \rangle$ as a factor. Then $O = P Sq^{2^{j+1}}$ has a term containing $\langle j - 1, e_1 \rangle \langle j - 1, e_2 \rangle$. The only way that this might be cancelled is if $M = \langle j - 1, e_3 \rangle \langle j, e_1 \rangle \langle j, e_2 \rangle M'$ (and hence $j > i$), and P also has a term $\langle j - 1, e_1 \rangle \langle j, e_2 \rangle \langle j, e_3 \rangle M'$ but does not have $\langle j - 1, e_2 \rangle \langle j, e_1 \rangle \langle j, e_3 \rangle M'$ as a term. Then $P Sq^{2^j} \neq 0$. Therefore for all $j > i$ and all monomials M in P , M has at most one factor with first component j .

Suppose P has a monomial $M = \langle j + 1, e \rangle M'$ with no first components j . Then $P Sq^{2^{j+1}}$ contains a term which cannot be cancelled. Therefore the first components of any monomial of P form a string of consecutive integers, and if the smallest of these is $j > i$, then $P Sq^{2^j} \neq 0$, so that the string must begin with i .

If $M = \langle j-1, e \rangle \langle j, e \rangle M' \in P$, then $M \text{Sq}^{2^j} = \langle j, e+1 \rangle M'$, which cannot be cancelled. Therefore M cannot have consecutive second components equal. If $M = \langle i, e_0 \rangle \cdots \langle i+k, e_k \rangle \in P$ and τ is the permutation which transposes $l-1$ and l , then consideration of Sq^{2^l} shows that P also contains $\langle i, e_{\tau(0)} \rangle \cdots \langle i+k, e_{\tau(k)} \rangle$. Since all permutations σ can be written as a product of such τ , P must contain $S_i(e_0, \dots, e_k)$.

Both Theorems 5 and 3 follow from the observation that $\xi_m^{2^r} \chi \text{Sq}^{2^j} = \xi_m^{2^r} \text{Sq}^{2^j}$ for all e, j , and m , and so the calculation of $\bigcap_{j \neq i} \ker R \chi \text{Sq}^{2^j} (= N_i^*)$ is the same as the above calculation of $\bigcap_{j \neq i} \ker R \text{Sq}^{2^j}$. To prove this observation first note that $\xi_i = \xi_i \text{Sq} \chi \text{Sq} = (\xi_i + \xi_{i-1}) \chi \text{Sq}$, from which $\xi_i \chi \text{Sq} = \sum_{m=0}^i \xi_m$ follows by induction. Thus $\xi_i^{2^r} \chi \text{Sq} = \sum_{m=0}^i \xi_m^{2^r}$, and the observation follows since $|\xi_i^{2^r}| - |\xi_m^{2^r}|$ is a 2-power if and only if $m = i-1$.

PROOF OF THEOREM 4. (ii) If $\text{Sq}^a x^j = x^k$, then $\alpha(k) \leq \alpha(j)$. Therefore $x^k \in L_i$ implies $\alpha(k) \leq i$. Suppose k is minimal with $\alpha(k) \leq i$ and $x^k \notin L_i$. There exists j such that $2^{j+1} \in k$ and $2^j \notin k$. Consideration of x^{k-2^j} yields a contradiction.

(i) $\text{Sq}^{2^j} x^{2^{i+1}-1} = 0$ if $j > i$ and, by (ii), $\text{Sq}^{2^j} x^{2^{i+1}-1} \in L_i$ if $j < i$. Thus $\theta \mapsto \theta x^{2^{i+1}-1}$ defines a surjection $\Sigma^{2^{i+1}-1} N_i \rightarrow L_{i+1}/L_i$. We show it injective by showing

$$\dim((\Sigma^{2^{i+1}-1} \chi N_i^*)_d) = \begin{cases} 1, & \alpha(d) = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

$|S_i(e_0, \dots, e_k)| + 2^{i+1} - 1 = 2^{i+k+2} - 1 - (2^{e_0} + \cdots + 2^{e_k})$. Since $e_k \leq i+k$, every $2^{f_0} + \cdots + 2^{f_i}$ with $f_0 < \cdots < f_i$ can be uniquely expressed in this way by $k = f_i - i - 1$ and $\{f_j\} = \{0, \dots, i+k+1\} - \{e_j\}$. $\square$

Finally Theorem 2 follows from Theorem 4 since $a_{l_0 0} \cdots a_{l_i i} x^{2^{i+1}-1} = x^{2^{l_0} + \cdots + 2^{l_i}}$. This is proved inductively:

$$a_{l_j j} \cdots a_{l_i i} x^{2^{i+1}-1} = x^{2^j + \cdots + 2^i + 2^j - 1}.$$

REFERENCES

1. D. W. Anderson, E. H. Brown and F. P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math. **86** (1967), 271–298.
2. D. M. Davis, *On the Segal conjecture for $Z_2 \times Z_2$* , Proc. Amer. Math. Soc. **82** (1981), 619–622.
3. J. W. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150–171.

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015