PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

AN INVARIANT FOR CONTINUOUS FACTORS OF MARKOV SHIFTS

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ABSTRACT. Let Σ_A and Σ_B be subshifts of finite type with Markov measures (p, P) and (q, Q). It is shown that if there is a continuous onto measure-preserving factor map from Σ_A to Σ_B, then the block of the Jordan form of Q with nonzero eigenvalues is a principal submatrix of the Jordan form of P. If Σ_A and Σ_B are irreducible with the same topological entropy, then the same relationship holds for the matrices A and B. As a consequence, $\zeta_B(t)/\zeta_A(t)$, the ratio of the zeta functions, is a polynomial. From this it is possible to construct a pair of equal-entropy subshifts of finite type that have no common equal-entropy continuous factor of finite type, and a strictly sofic system that cannot have an equal-entropy subshift of finite type as a continuous factor.

1. Introduction. Let A be an $l \times l$ matrix of 0's and 1's. The subshift of finite type, Σ_A, determined by A is the closed invariant subspace of $\{1, \ldots, l\}^{\mathbf{Z}}$ consisting of all $x = \ldots x_{-1}x_0x_1 \ldots$ such that $A_{x_i x_{i+1}} = 1$ for all i, together with the shift transformation [9]. A Markov measure is defined on this space by a pair (p, P), where P is a stochastic matrix compatible with A (i.e., positive, row-sum 1, and $P_{ij} > 0$ exactly when $A_{ij} = 1$) and p is a probability vector with $pP = p$ [9].

The following dynamical properties of subshifts of finite type will be used. A subshift of finite type is topologically transitive when its transition matrix is irreducible [9]. The irreducibility of the transition matrix is also the condition needed for erogodicity with respect to any Markov measure [9]. The zeta function of a dynamical system is $\zeta(t) = \exp[\Sigma_{n=1}^{\infty}(N_n)t_n/n]$, where N_n is the number of points fixed by the nth power of the transformation. For a subshift of finite type, Σ_A, this has the form $\zeta_A(t) = [t^l C_A(1/t)]^{-1}$, where $C_A(x)$ is the characteristic polynomial of A [2]. The topological entropy of an irreducible subshift of finite type is $\log \lambda$, where λ is the largest positive real eigenvalue [9], and the measure-theoretic entropy with respect to a Markov measure (p, P) is $-\Sigma_{ij} p_i P_{ij} \log P_{ij}$ [9]. An irreducible subshift of finite type has a unique measure of maximal entropy [9], which is a Markov measure whose matrix has the form $P = \frac{1}{\lambda} R^{-1}AR$ (R a diagonal matrix with strictly positive diagonal entries). Any continuous onto finite-to-one factor map between two subshifts of finite type will carry the measure of maximal entropy of one to the measure of maximal entropy of the other [3]. The Curtis-Hedlund-Lyndon theorem [6] asserts that any continuous factor map between subshifts of finite type is a block map composed with some power of the shift. This

Received by the editors March 10, 1981.
1980 *Mathematics Subject Classification.* Primary 58F20.
Key words and phrases. Subshift of finite type, Markov shift, zeta function, factor maps.

0002-9939/81/0000-0581/$02.00

means that by going to a higher block presentation of the domain shift, any continuous factor map can be expressed as a one-block map [1].

2. P's and Q's.

THEOREM. *If Σ_A and Σ_B are subshifts of finite type and if there is a continuous onto factor map between them that takes (p, P) to (q, Q), then the block of the Jordan form of Q with nonzero eigenvalues is a principal submatrix of the Jordan form of P.*

PROOF. We may assume that the factor map $\phi: \Sigma_A \to \Sigma_B$ is a one-block map [10]. Take L_A to be the alphabet of Σ_A and L_B the alphabet of Σ_B. R will be the relation matrix that represents the equivalence relation defined on L_A by ϕ: for $i \in L_A$, $a \in L_B$

$$R_{ia} = \begin{cases} 1 & \text{if } \phi(i) = a, \\ 0 & \text{otherwise.} \end{cases}$$

A time-zero cylinder set is defined by $[a_1, \ldots, a_n] = \{x: x_0 = a_1, \ldots, x_{n-1} = a_n\}$. The inverse image of a time-zero cylinder set in Σ_B is a finite union of time-zero cylinder sets in Σ_A, of the same length. For any $[a_1, \ldots, a_n] \subseteq \Sigma_B$ define $u^{[a_1, \ldots, a_n]} \in \mathbf{R}^{|L_A|}$ and $v^{[a_1, \ldots, a_n]} \in \mathbf{R}^{|L_B|}$ by $(u^{[a_1, \ldots, a_n]})_i = \sum p_{i_1} P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$, where the sum is taken over all $[i_1, \ldots, i_n]$ in $\phi^{-1}[a_1, \ldots, a_n]$ such that $i_n = i$,

$$(v^{[a_1, \ldots, a_n]})_a = \begin{cases} q_{a_1} Q_{a_1 a_2} \cdots Q_{a_{n-1} a_n} & \text{if } a_n = a, \\ 0 & \text{otherwise.} \end{cases}$$

Let U be the collection of all $u^{[a_1, \ldots, a_n]}$ and V that of all $v^{[a_1, \ldots, a_n]}$. V generates all of $\mathbf{R}^{|L_B|}$; let $\mathfrak{U}$ be the subspace of $\mathbf{R}^{|L_A|}$ generated by U. A computation using the measure-preserving property of ϕ shows the diagram

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{P} & \mathfrak{U} \\ R\downarrow & & \downarrow R \\ \mathbf{R}^{|L_B|} & \xrightarrow{Q} & \mathbf{R}^{|L_B|} \end{array}$$

commutes, where the matrices operate by left multiplication, i.e. $xPR = xRQ$ for all $x \in \mathfrak{U}$. Since R has rank $|L_B|$, we have the desired result. Notice that this linear algebra situation is equivalent to the existence of such a factor map.

COROLLARY A. *If Σ_A, Σ_B are irreducible subshifts of finite type and Σ_B is a continuous finite-to-one factor of Σ_A, then the block of the Jordan form of B with nonzero eigenvalues is a principal submatrix of the Jordan form of A. In particular, if $\zeta_A(t)$ and $\zeta_B(t)$ are the zeta functions, then $\zeta_B(t)/\zeta_A(t)$ is a polynomial.*

PROOF. Σ_A and Σ_B have the same topological entropy, $\log \lambda$. The matrices P, Q for the measures of maximal entropy are $P = \frac{1}{\lambda} R^{-1}AR$ and $Q = \frac{1}{\lambda} S^{-1}BS$, where R, S are the appropriate diagonal matrices of full rank. The desired result is obtained by applying the thoerem. Recalling that $\zeta_A(t) = (t^{|L_A|} C_A(1/t))^{-1}$, we have the observation about the ratio of the zeta functions. We also have a completely topological proof of this corollary, and M. Nasu [8] has proved the fact about the ratio of the zeta functions using graph-theoretic techniques.

COROLLARY B. *There exist equal-entropy mixing* [**1**] *subshifts of finite type that have no common equal-entropy continuous factor. This is in contrast to the Adler-Marcus Theorem, which asserts that any such shifts have a common equal-entropy continuous extension* [**1**].

PROOF. Take

$$A = \begin{bmatrix} 001 \\ 101 \\ 010 \end{bmatrix}, \qquad B = \begin{bmatrix} 00001 \\ 10000 \\ 01000 \\ 00100 \\ 00011 \end{bmatrix},$$

then $C_A(x) = x^3 - x - 1$, which is irreducible, and $C_B(x) = x^5 - x^4 - 1 = (x^3 - x - 1)(x^2 - x + 1)$. These are both mixing and have the same entropy. Since $C_A(x)$ is irreducible over $\mathbf{Z}$, any continuous finite-to-one factor of Σ_A must have the same zeta function. Σ_B has a fixed point, so any factor of it must have a fixed point. There is no subshift of finite type that meets both of these requirements.

COROLLARY C. *There exists a mixing strictly sofic system* [**11**] *that has no equal-entropy subshift of finite type as a continuous factor. This should be compared to the fact that any sofic system is a continuous equal-entropy factor of a subshift of finite type* [**4**].

PROOF. Begin with Σ_A as in the previous corollary. Obtain a strictly sofic system of the same entropy by identifying the pair of two-blocks [2, 3] and [3, 2]. This sofic system has a fixed point. Any subshift of finite type that is an equal-entropy continuous factor of this system is also one for Σ_A. We already know there is no such shift. This construction was noticed by Brian Marcus.

COROLLARY D. *Any equal-entropy continuous factor of the full shift which is a subshift of finite type is shift equivalent* (*in the sense of Williams* [**12**]) *to the same full shift. This was previously proved in* [**7**].

PROOF. Any equal-entropy continuous factor of the full n-shift that is a subshift of finite type has the zeta function $(1 - nt)^{-1}$. R. Williams [**12**] has shown that any subshift of finite type with this zeta function is shift equivalent to the full n-shift. It is possible to deduce Corollaries B and C from the work of J. Cuntz and W. Krieger [**5**].

I would like to thank my advisor, Brian Marcus, for getting me started and keeping me going on these questions.

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