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INVOLUTIONS WITH FIXED POINT SET OF CONSTANT CODIMENSION

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ABSTRACT. The cobordism classes of manifolds admitting involutions with fixed point set of codimension 5 are determined by means of Stiefel-Whitney classes.

1. Introduction. Let $\mathfrak{N}_n$ be the group of nonoriented cobordism classes of n-dimensional smooth manifolds and let J_n^k be its subset consisting of the classes which are represented by manifolds admitting smooth involutions with fixed point set of constant codimension k. J_n^k is a subgroup of $\mathfrak{N}_n$ and $J_*^k = \Sigma_{n=k}^{\infty} J_n^k$ is an ideal of the nonoriented cobordism ring $\mathfrak{N}_* = \Sigma_{n=0}^{\infty} \mathfrak{N}_n$. Capobianco [1] proved the following results:

PROPOSITION 1. *J_n^3 is the set of classes α in $\mathfrak{N}_n$ with $W_1^j W_{n-j}(\alpha) = W_1^{i-5} W_{n-i} s_5(\alpha) = 0$, for each i, j, $0 \leqslant j \leqslant n$, $5 \leqslant i \leqslant n$.*

PROPOSITION 2. $J_*^5 \subset J_*^3$.

In this note, we shall prove

THEOREM. *J_n^5 is the set of classes α in J_*^3 with $W_1^{n-8} W_2^4(\alpha) = W_1^{n-9} W_2^3 W_3(\alpha) = W_1^{n-10} W_2^2 W_3^2(\alpha) = W_1^{n-11} W_2 W_3^3(\alpha) = W_1^{n-12} W_3^4(\alpha) = 0$.*

2. Characteristic numbers of classes in J_*^5. Let $\xi \to V$ be a smooth k-plane bundle over a closed smooth manifold V and let π: $RP(\xi) \to V$ be the associated projective. space bundle. Denote by a the characteristic class of the canonical line bundle $\lambda \to RP(\xi)$. Then by [2, §21], $H^*(RP(\xi); Z_2)$ is the free $H^*(V; Z_2)$-module via π^* on the classes $1, a, \ldots, a^{k-1}$, subject to the relation $\Sigma_{j=0}^k a^{k-j} \pi^*(v_j) = 0$, where v_j is the jth Whitney class of ξ. The total Stiefel-Whitney class of $RP(\xi)$ is given by

$$W(RP(\xi)) = \pi^*(W(V))\left(\sum_{j=0}^{k} (1 + a)^{k-j} \pi^*(v_j)\right).$$

Now suppose that a class α is represented by a manifold M^n admitting an involution with fixed point set F of codimension k. Let q: $\nu \to F$ be the normal bundle. Then by [2, (22.2)], α is the class of $RP(\nu \oplus R)$, which is the total space of the projective space bundle associated to p: $\nu \oplus R \to F$. Let e, resp. c, be the characteristic class of the canonical line bundle $\lambda \to RP(\nu \oplus R)$, resp. $\lambda \to RP(\nu)$.

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Then we have

PROPOSITION 3. *For any* $x \in H^j(F; Z_2)$, $0 \leq j \leq n-k$,

$$\langle p^*(x)e^{n-j}, [RP(\nu \oplus R)]\rangle = \langle q^*(x)c^{n-1-j}, [RP(\nu)]\rangle.$$

The proof can be found in [5]. It follows from [2, §25]

PROPOSITION 4. *If* $q^*(x)c^{n-1-j}$ *represents a characteristic class of* $\lambda \to RP(\nu)$, *then* $\langle p^*(x)e^{n-j}, [RP(\nu \oplus R)]\rangle = 0$.

Let us apply these facts to the case of constant codimension 5.

LEMMA 5. *If* $\alpha \in J_*^5$, *then* $W_1^{n-8}W_2^4(\alpha) = W_1^{n-9}W_2^3W_3(\alpha) = W_1^{n-10}W_2^2W_3^2(\alpha) = W_1^{n-11}W_2W_3^3(\alpha) = W_1^{n-12}W_3^4(\alpha) = 0$.

PROOF. Denote $W_j = W_j(RP(\nu \oplus R))$, $W_j' = W_j(RP(\nu))$ and let v_j, resp. w_j, be the jth Whitney class of ν, resp. F. By Propositions 3 and 4,

$$\begin{cases} W_1^{n-8}W_2^4 = p^*(w_1+v_1)^{n-8}e^8, \\ (W_1'+c)^{n-8}c^7 = q^*(w_1+v_1)^{n-8}c^7 \end{cases}$$

give $W_1^{n-8}W_2^4(\alpha) = 0$. The other relations can be obtained by the following computations:

$$\begin{cases} W_1^{n-9}W_2^3W_3 = p^*\{(w_1+v_1)^{n-9}w_1\}e^8, \\ (W_1'+c)^{n-9}\{(W_2'+W_1'c)(W_3'+W_2'c)c^3 + (W_1'+c)(W_2'+W_1'c)^2c^3 \\ \qquad\qquad + (W_1'+c)c^7 + (W_2'+W_1'c+c^2)^4\} \\ \qquad = q^*\{(w_1+v_1)^{n-9}w_1\}c^7, \end{cases}$$

$$\begin{cases} W_1^{n-10}W_2^2W_3^2 = p^*\{(w_1+v_1)^{n-10}w_1^2\}e^8, \\ (W_1'+c)^{n-10}\{W_1'^2W_2'^2c^3 + W_3'^2c^3 + W_2'^4c\} = q^*\{(w_1+v_1)^{n-10}w_1^2\}c^7, \end{cases}$$

$$\begin{cases} W_1^{n-11}W_2W_3^3 = p^*\{(w_1+v_1)^{n-11}w_1^3\}e^8, \\ (W_1'+c)^{n-11}\{W_2'^2W_3'^2 + W_2'^3W_3'c + (W_2'^4 + W_1'^2W_3'^2 + W_1'W_2'^2W_3')c^2 \\ \qquad\qquad + (W_1'W_2'^3 + W_1'W_3'^2)c^3\} \\ \qquad = q^*\{(w_1+v_1)^{n-11}w_1^3\}c^7, \end{cases}$$

$$\begin{cases} W_1^{n-12}W_3^4 = p^*\{(w_1+v_1)^{n-12}w_1^4\}e^8, \\ (W_1'+c)^{n-12}W_2'^4c^3 = q^*\{(w_1+v_1)^{n-12}w_1^4\}c^7. \end{cases}$$

3. A system of generators of J_*^5. As is well known, $\mathfrak{N}_*$ is a graded polynomial algebra over Z_2 with one generator in each dimension n which is not of the form $2^r - 1$. We shall choose a suitable system of generators of $\mathfrak{N}_*$ for our purpose. Let $(n_1, n_2, \ldots, n_{2k})$ be a $2k$-tuple of nonnegative integers with $n_1 + n_2 + \cdots + n_{2k} = n - 2k + 1$. We denote by $RP(n_1, n_2, \ldots, n_{2k})$ the projective space bundle associated to the bundle $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_{2k}$ over $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_{2k})$, where λ_i $(i = 1, 2, \ldots, 2k)$ is the pull-back of the canonical line bundle over the ith factor. Stong [4, Lemma 3.4] proved that

$RP(n_1, n_2, \ldots, n_{2k})$ belongs to J_n^k and is indecomposable in $\mathfrak{N}_*$ if and only if

$$\binom{n-1}{n_1} + \binom{n-1}{n_2} + \cdots + \binom{n-1}{n_{2k}}$$

is odd. First, we shall show

LEMMA 6. *For each $n \geq 13$, not of the form 2^r or $2^r - 1$, there exists a generator $u_n \in J_n^5$ which is indecomposable in $\mathfrak{N}_*$.*

PROOF. If $\binom{n-1}{n-9} \equiv 0 \bmod 2$, $RP(n-9, 0, \ldots, 0)$ (9 zeroes) is indecomposable in $\mathfrak{N}_*$. Consider the case $\binom{n-1}{n-9} \equiv 1 \bmod 2$. Let $n - 1 = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$, $r_1 > r_2 > \cdots > r_t \geq 0$. Since $\binom{n-1}{n-9} = \binom{n-1}{8}$, $\{r_1, r_2, \ldots, r_t\}$ contains 3. When $\{r_1, r_2, \ldots, r_t\}$ does not contain 1, $RP(n-11, 1, 1, 0, \ldots, 0)$ (7 zeroes) is indecomposable in $\mathfrak{N}_*$. When $\{r_1, r_2, \ldots, r_t\}$ contains 1 but does not contain 2, we can choose $RP(n-13, 2, 2, 0, \ldots, 0)$ (7 zeroes) as an indecomposable generator of $\mathfrak{N}_*$. Finally, suppose that $\{r_1, r_2, \ldots, r_t\}$ contains both 1 and 2. Since $n-1$ is not of the form $2^r - 1$ or $2^r - 2$, there exists a number i such that $r_i > r_{i+1} + 1$. Then, $RP(2^{r_1} + \cdots + 2^{r_i} - 2, 2^{r_{i+1}} + \cdots + 2^{r_t} - 14, 8, 0, \ldots, 0)$ (7 zeroes) is indecomposable in $\mathfrak{N}_*$.

Let x_2 be the class of $RP(2)$ and let x_{2^n} be the class of $RP(2^n) \cup RP(2)^n$ for $n > 1$. Denote by y_n $(n = 5, 6)$ the class of $RP(n-3, 0, 0, 0)$ and by z_n $(n = 9, 10, 12)$ the class of $RP(n-5, 0, 0, 0, 0, 0)$. Furthermore, by [3, §7, Remark] we know that there exists a class z_{11} of an indecomposable manifold which belongs to J_{11}^5. Thus we have

LEMMA 7. *$\mathfrak{N}_*$ is a polynomial algebra over Z_2 with the system of generators*: $\{x_{2^n}$ $(n = 1, 2, \ldots)$, u_n $(n \geq 13, n \neq 2^r, 2^r - 1)$, $y_5, y_6, z_9, z_{10}, z_{11}, z_{12}\}$.

Now we shall go into J_n^5. By direct computations as in Lemma 5, we have

LEMMA 8. *If $n \leq 10$ or $n = 12$, then $s_n(\alpha) = 0$ for any $\alpha \in J_n^5$.*

Moreover, we have

LEMMA 9. *Let $n = 2^s$, $s \geq 4$. Then J_n^5 contains a class α such that $s_{2^{s-1}, 2^{s-1}}(\alpha) \equiv 1$* mod 2.

PROOF. For $s = 4$, $RP(7, 0, \ldots, 0)$ (9 zeroes) is as required. For $s > 4$,

$$RP(2^{s-3} - 2, s^{s-3} - 1, \ldots, 2^{s-3} - 1, 0, 0)$$

is as required.

Let us observe monomials of the generators for $\mathfrak{N}_*$. First, notice that $J_*^3 \subset J_*^2$ follows from [4] and Proposition 1. By their definitions, $y_5, y_6 \in J_*^2$ and $z_9, z_{10}, z_{12} \in J_*^3$. Furthermore, Proposition 1 shows $y_5 y_6$, $x_4^2 \in J_*^3$ and Lemma 9 shows $x_{2^r}^2 \in J_*^5$ for $r \geq 3$. Clearly, $y_5^2 \in J_*^5$. Now consider y_6^2. By the examination of the characteristic numbers, we can see that y_6^2 is the class $x_2 y_5^2 + \{RP(3, 2, 2, 0, 0, 0)\}$. Recall that $RP(\lambda) = RP(3, 2, 2, 0, 0, 0)$ is the projective space bundle over $M = RP(3) \times RP(2)^2 \times RP(0)^3$ associated to $\lambda = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_6 \to M$. An involution of M, given by $(a, b_1, b_2, c_1, c_2, c_3) \to (a, b_2, b_1, c_1, c_2, c_3)$, induces a fiber preserving involution T of $RP(\lambda)$; i.e., we can define an involution T of $RP(\lambda)$ by

$$T(u_1, u_2, u_3, u_4, u_5, u_6) = (-u_1, -u_3, -u_2, -u_4, u_5, u_6).$$

It is easy to see that all the components of the fixed point set of T are of codimension 5. Therefore $y_6^2 \in J_*^5$. Referring to the results of [1], we can show that J_*^5 contains all monomials of generators for J_*^3 except those of the form

$$y_5 y_6 x(m), \quad z_9 x(m), \quad x_4^2 x(m), \quad z_{10} x(m), \quad z_{12} x(m).$$

Here, $x(m)$ is the class of $RP(2^{r_1}) \times RP(2^{r_2}) \times \cdots \times RP(2^{r_t})$ for $m = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$, $r_1 > r_2 > \cdots > r_t > 0$. By straightforward calculation, we have the tables of characteristic numbers.

	$z_9 x(n-9)$	$y_5 y_6 x(n-11)$
$W_1^{n-9} W_2^3 W_3$	1	0
$W_1^{n-11} W_2 W_3^3$		1

	$x_4^2 x(n-8)$	$z_{10} x(n-10)$	$z_{12} x(n-12)$
$W_1^{n-8} W_2^4$	1	0 $n \equiv 0, 4\ (8)$ 1 $n \equiv 2, 6\ (8)$	0 $n \equiv 0, 2\ (8)$ 1 $n \equiv 4, 6\ (8)$
$W_1^{n-10} W_2^2 W_3^2$	1	1 $n \equiv 0, 4\ (8)$ 0 $n \equiv 2, 6\ (8)$	1 $n \equiv 0, 6\ (8)$ 0 $n \equiv 2, 4\ (8)$
$W_1^{n-12} W_3^4$	1	0 $n \equiv 0, 4\ (8)$ 1 $n \equiv 2, 6\ (8)$	1 $n \equiv 0, 2\ (8)$ 0 $n \equiv 4, 6\ (8)$

Using these, together with Lemmas 5, 7 and 8, we can attain our theorem immediately.

REMARK. As a corollary, we can show $J_*^{2k+1} \subset J_*^5$ for every integer $k \geqslant 3$.

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