PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

A CHARACTERIZATION OF MANIFOLD DECOMPOSITIONS SATISFYING THE DISJOINT TRIPLES PROPERTY

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ABSTRACT. A metric space X satisfies the Disjoint Triples Property (DD_3) if maps f_1, f_2 and f_3 from B^2 into X are approximable by maps $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$ with $\bigcap_{i=1}^{3} \tilde{f}_i(B^2) = \varnothing$. Those CE decompositions of manifolds satisfying DD_3 and yielding finite-dimensional nonmanifold decomposition spaces are shown to be precisely those intrinsically 0-dimensional decompositions which yield nonshrinkable null cellular decompositions under amalgamation. This characterization results in another proof of the fact that $E^n/G \times E^1$ is secretly 0-dimensional where G is a CE usc decomposition of E^n, $n > 4$, with E^n/G finite dimensional.

0. Introduction. Recent work of Cannon [C], Edwards [Ed] and Quinn [Q] characterizes topological n-manifolds, $n \geqslant 5$, in terms of a simple general position property, the Disjoint Disks Property (DDP). Daverman introduced a property closely related to the DDP, the Disjoint Triples Property (hereafter referred to as DD_3), and showed that certain decompositions satisfying DD_3 can be amalgamated so as to yield nonshrinkable null cellular decompositions. See [D1].

We show that finite-dimensional nonshrinkable decompositions satisfying DD_3 can be characterized as those intrinsically 0-dimensional decompositions having amalgamations as above. Thus, the class of finite-dimensional decompositions satisfying DD_3 but not DDP has the minimal amount of complexity required to yield nonmanifold decompositions in the sense that nonshrinkable null cellular decompositions form a prototype for this class.

It is also shown that $E^n/G \times E^1$ satisfies DD_3 where G is any CE uscd of E^n, $n \geqslant 4$. This result, combined with the above characterization, gives another proof of the fact that $E^n/G \times E^1$ is secretly 0-dimensional when E^n/G is finite dimensional [D3, p. 133].

This paper contains results presented in my thesis [G1] completed under the supervision of J. W. Cannon.

1. Notation and preliminaries. We will be considering cell-like (CE) upper semicontinuous decompositions (uscd) G of n-manifolds M. If G is such a decomposition, π or π_G represents the natural quotient map from M onto M/G, H_G represents the set consisting of the nondegenerate elements of G, and N_G represents

Received by the editors November 14, 1980.

1980 *Mathematics Subject Classification.* Primary 54F65, 54B15, 57N60, 57P95; Secondary 57N15, 54B10, 54N75.

Key words and phrases. Cell-like decomposition, cellular set, disjoint disks property, disjoint triples property, intrinsic dimension, null sequence, secret dimension, shrinkable.

0002-9939/81/0000-0583/$02.50

the union of these elements. Given a closed subset A of M/G, $M/\pi^{-1}(A) \equiv M/G'$ where G' is the decomposition having as elements the singletons of $M - \pi^{-1}(A)$ together with the elements of $\{\pi^{-1}(a) | a \in A\}$.

If p is a CE map from M onto X, $N_p \equiv N_L$ where L is the decomposition of M with elements $\{p^{-1}(x) | x \in X\}$. The map is said to be 1-1 over $A \subset X$ if $p | p^{-1}(A)$ is 1-1. The metric on a space X will be denoted by ρ or ρ_x. The space $\mathbb{S}$ of maps from a compact space X into a complete separable metric space Y will be given the topology induced by the metric $\rho_{\mathbb{S}}(f, g) \equiv \sup_{x \in X} \rho_Y(f(x), g(x))$. By [**K**, p. 93], $(\mathbb{S}, \rho_{\mathbb{S}})$ is a complete separable metric space. To say that a map f from X into Y is *approximable* by maps g will mean, using the above notation, that for each $\varepsilon > 0$ there exists a map g with $\rho_{\mathbb{S}}(f, g) < \varepsilon$.

DEFINITION. Let G be a CE uscd of an n-manifold M and f a map from B^2 into M/G. Then maps F: $B^2 \to M$ such that $\pi \circ F$ is an approximation to f are called *approximate lifts* of f.

By [**L**, p. 506], f is always approximable by maps $\pi \circ F$ where F is an *approximate lift* of f.

DEFINITION. Let G be a CE uscd of an n-manifold M. Then G is *secretly d-dimensional* if π is approximable by CE maps p from M onto M/G with the dimension of $p(Np) \leqslant d$. G is said to be *intrinsically d-dimensional* if it is secretly d-dimensional, but is not secretly $(d-1)$-dimensional.

DEFINITION. A metric space X has property DD_K, $2 \leqslant K < \infty$, if maps f_i: $B^2 \to X$, $1 \leqslant i \leqslant K$, are approximable by maps $\tilde{f}_i$: $B^2 \to X$, $1 \leqslant i \leqslant K$, with $\bigcap_{i=1}^{K} \tilde{f}_i(B^2) = \varnothing$.

REMARK 1. Property DDP is the same as DD_2. The Disjoint Triples Property is the same as DD_3. A decomposition G of M is said to satisfy DD_K if M/G satisfies DD_K.

DEFINITION. Let G be a CE uscd of an n-manifold M and let $\varepsilon > 0$ be fixed. Then $\mathcal{A}$ is said to be an *ε-amalgamation* of G if

(i) $\mathcal{A}$ is a CE uscd of M,

(ii) each $g \in G$ is contained in some $a \in \mathcal{A}$, and

(iii) for each $x \in M/\mathcal{A}$, $\pi_G \circ \pi_{\mathcal{A}}^{-1}(x)$ is of diameter $< \varepsilon$.

REMARK 2. The space M/A can be thought of as a CE decomposition of M/G.

The following technical lemma can be proved using standard properties of CE maps.

LEMMA 1. *Let G be a CE uscd of an n-manifold M, $n \geqslant 5$, and A closed in M/G. If pairs of maps (f_1, f_2): $B^2 \to M/G$ are approximable by pairs $(\tilde{f}_1, \tilde{f}_2)$ with $\tilde{f}_1(B^2) \cap \tilde{f}_2(B^2) \cap A = \varnothing$, then $M/\pi^{-1}(A)$ satisfies DDP.*

2. Spaces that satisfy DD_3. In this section, sufficient conditions for a decomposition to satisfy DD_3 are given and results leading to the characterization of the next section are presented.

PROPOSITION 1. *Let G be a CE uscd of an n-manifold M, $n \geqslant 4$, such that H_G consists of a countable number of cellular sets. Then M/G satisfies DD_3.*

PROOF. Choose $\varepsilon > 0$ and maps f_e: $B^2 \to M/G$, $1 \leqslant e \leqslant 3$. Choose approximate lifts F_1 and F_2 to f_1 and f_2 so that $F_1(B^2) \cap F_2(B^2)$ consists of (at most) a finite number of points, and so that $\tilde{f}_e \equiv \pi \circ F_e$, $e = 1$ or 2, is within ε of f_e. Then $\tilde{f}_1(B^2) \cap \tilde{f}_2(B^2) \equiv A$ consists of at most countably many points in M/G.

Since G is cellular, an ε approximation $\tilde{f}_3$ to f_3 can be found so that $\tilde{f}_3(B^2) \cap A = \varnothing$. The required approximations are $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$.

The examples of [**D1**, **W** and **DW**] are of nonshrinkable decompositions satisfying the hypotheses of Proposition 1, thus showing that there exist nonmanifold spaces satisfying DD_3.

PROPOSITION 2. *A complete separable metric, ANR X, satisfies DD_3 if and only if each map f from B^2 into X is approximable by maps $\tilde{f}$ such that $\{\tilde{f}^{-1}(x)\}$ has cardinality less than* 3 *for each x in X.*

PROOF. The proof of the forward implication is modeled after the argument presented in [**C**, Theorem 2.1]. Let $\{(P_i(1), P_i(2), P_i(3))\}$, $1 \leqslant i < \infty$, be a countable collection of triples of subdiscs of B^2 chosen so that:

(1) for each i, $P_i(j) \cap P_i(k) = \varnothing$ for $j \neq k$; and

(2) given distinct points x_1, x_2 and x_3 in B^2, there exists an i such that $x_e \in P_i(e)$, $1 \leqslant e \leqslant 3$.

Choose $\varepsilon > 0$ and f: $B^2 \to X$. Since X satisfies DD_3, $\theta_i \equiv \{h\colon B^2 \to X \mid \bigcap_{e=1}^{3} h(P_i(e)) = \varnothing\}$ is dense and open in the space of all maps from B^2 into X. It follows from the Baire Category Theorem [**CV**, p. 89] that there exists a map $\tilde{f}$ in $\bigcap_{i=1}^{\infty} \theta_i$ with $\rho(f, \tilde{f}) < \varepsilon$. The map $\tilde{f}$ is the required approximation to f.

For the reverse implication, choose $\varepsilon > 0$ and maps f_e: $B^2 \to X$, $1 \leqslant i \leqslant 3$. Choose arcs h joining $f_1(\partial B^2)$ to $f_2(\partial B^2)$ and l joining $f_2(\partial B^2)$ to $f_3(\partial B^2)$. Then $(\bigcup_{e=1}^{3} f_e(B^2)) \cup h \cup l$ can be viewed as the image of a map F from B^2 into X, with subdiscs D_e, $1 \leqslant e \leqslant 3$, of B^2 so that $F|D_e$ is an $\varepsilon/3$ approximation to f_e.

By assumption, F can be $\varepsilon/3$ approximated by a map $\tilde{f}$ so that each point of X has less than 3 preimages under $\tilde{f}$. The required approximations to f_e, $1 \leqslant e \leqslant 3$, are the $\tilde{f}|D_e$.

COROLLARY 1. *Let G be a CE uscd of an n-manifold M, $n \geqslant 5$, so that M/G is an ANR and satisfies DD_3. Then G is secretly* 0*-dimensional.*

PROOF. Let $\{(F_i, H_i)\}$, $1 \leqslant i < \infty$, be a countable dense subset of the space of pairs of maps from B^2 into M/G. The argument used in the proof of Proposition 2 allows one to require that, for each i, $F_i^{-1}(x) \cup H_i^{-1}(x)$ has cardinality less than 3 for each $x \in M/G$.

Thus, each $H_i(B^2)$ is the, at most, 2-to-1 image of a 2-dimensional space, and so has dimension $\leqslant 3$ [**En**, p. 134]. This allows one to further require that, for each i, $F_i^{-1}(H_i(B^2))$ is 0-dimensional [**C**, Theorem 2.2]. Thus, $F_i(B^2) \cap H_i(B^2)$ is the 1-1 image of a 0-dimensional set and so has dimension 0 [**En**, p. 134]. Let $A = \bigcup_{i=1}^{\infty}[F_i(B^2) \cap H_i(B^2)]$.

The set A is a 0-dimensional $F\sigma$, and so by [**En**, p. 45], there exist a 0-dimensional G_δ, L, with $A \subset L$. Let C be any closed set in $M/G - L$. Lemma 1 implies that $M/\pi^{-1}(C)$ satisfies the DDP, so by **[Ed]**, p: $M \twoheadrightarrow M/\pi^{-1}(C)$ is approximable by homeomorphisms. By **[Ev**, p. 15], it follows that π: $M \to M/G$ is approximable by a CE map q that is 1-1 over $M/G - L$. It follows that $q(N_q)$ is contained in the 0-dimensional set L, and so G is secretly 0-dimensional.

PROPOSITION 3. *Let G be a CE uscd of an n-manifold M, $n \geq 4$. Then $(M/G \times E^1)$ satisfies DD_3.*

PROOF. Choose $\varepsilon > 0$ and maps f_e: $B^2 \to (M/G \times E^1)$, $1 \leq e \leq 3$. Let p be the projection of $M/G \times E^1$ onto M/G and q the projection onto E^1. Choose a triangulation T of B^2 so that for each $\sigma \in T$, and for $1 \leq e \leq 3$, there exist open sets $V_e(\sigma)$ and $W_e(\sigma)$ in M/G, and open intervals $K_e(\sigma)$ and $J_e(\sigma)$ in E^1 satisfying:

(1) $f_e(\sigma) \subset V_e(\sigma) \times K_e(\sigma) \subset W_e(\sigma) \times J_e(\sigma)$;

(2) $W_e(\sigma) \times J_e(\sigma)$ has diameter $< \varepsilon/3$; and

(3) $V_e(\sigma)$ contracts in $W_e(\sigma)$.

By **[D2**, Lemma 2.6], there exists $\varepsilon/3$ approximations h_e to f_e, $1 \leq e \leq 3$, so that

(4) $(p \circ h_e|T^1) \cap (p \circ h_d|T^1) = \varnothing$ for $e \neq d$, and

(5) for each σ in T, $h_e(\sigma) \subset V_e(\sigma) \times K_e(\sigma)$.

Let P_1, P_2 and P_3 be pairwise disjoint dense subsets of E^1. For each two cell $\sigma \in T$, let $C(\sigma)$ be a small interior collar on $\partial\sigma$. Let $N(\sigma)$ represent the interior boundary of $C(\sigma)$, $C = \bigcup_{\sigma \in T} C(\sigma)$, and $U = B^2 - C$.

Focus attention on a specific two cell σ. Define an approximation $\tilde{f}_e$ to h_e, $1 \leq e \leq 3$, as follows:

(6) $\tilde{f}_e|C(\sigma)$ is defined so that $p \circ \tilde{f}_e(C(\sigma)) = p \circ h_e(\partial\sigma)$;

(7) $q \circ \tilde{f}_e(N(\sigma)) \equiv a_e(\sigma)$, a point in $P_e \cap K_e(\sigma)$;

(8) $\tilde{f}_e|C(\sigma)$ is extended to all of σ using the contraction of $V_e(\sigma)$ in $W_e(\sigma)$ so that $q \circ \tilde{f}_e(\sigma - C(\sigma)) = a_e(\sigma)$.

Condition (6) guarantees that $[p(\tilde{f}_e(C))] \cap [p(\tilde{f}_d(C))] = \varnothing$ for $e \neq d$. Condition (8) guarantees that $[q(\tilde{f}_e(U))] \cap [q(\tilde{f}_d(U))] = \varnothing$ for $e \neq d$. It follows that $\bigcap_{e=1}^{3} \tilde{f}_e(B^2) = \varnothing$ and that $M/G \times E^1$ satisfies DD_3.

THEOREM 1 **[D3**, p. 133]. *Let G be as in Propositions 3 with M/G finite dimensional. Then $M/G \times E^1$ is secretly 0-dimensional.*

PROOF. This follows directly from Corollary 1 and Proposition 3.

3. The characterization. The following amalgamation lemma can be found in slightly different form in **[D1**, p. 173 and **Ed]**. It has shown its importance in decomposition theory, both in dimension 4 and in dimensions ≥ 5. See **[DP, DR** and **Ed]**.

AMALGAMATION LEMMA. *Let G be a CE uscd of an n-manifold M, $n \geq 3$, so that $\pi(N\pi)$ is 0-dimensional. Let $F = \bigcup_{i=1}^{\infty} F_i$ be an $F\sigma$ set in $M/G - \pi(N_G)$ of dimension $\leq n-2$. Then, for each $\varepsilon > 0$, there exists an ε-amalgamation $A(\varepsilon)$ of G such that*

(1) *$H_{A(\varepsilon)}$ consists of a null sequence and*

(2) *$N_{A(\varepsilon)} \cap \pi^{-1}(F) = \varnothing$.*

The next lemma follows from Bing's Shrinking Criterion [**B**]. For a proof, see [**D1**, p. 174 or **Ed**].

LEMMA 3. *Let G be a CE uscd of an n-manifold M, $n \geqslant 5$, such that for every $\varepsilon > 0$ there exists an ε-amalgamation $A(\varepsilon)$ of G with $M/A(\varepsilon) \cong M$. Then $M/G \cong M$.*

THEOREM 2 (CHARACTERIZATION THEOREM). *Let G be a CE uscd of an n-manifold M, $n \geqslant 5$, with M/G an ANR. Then M/G satisfies DD_3 but not DDP if and only if*

(1) G is intrinsically 0-dimensional, i.e. there exists a CE uscd G' of M with $M/G \cong M/G' \not\cong M$, and $\pi_{G'}(N_{G'})$ of dimension 0, and

(2) for each $\varepsilon > 0$, there exists an ε-amalgamation $A(\varepsilon)$ of G' with $H_{A(\varepsilon)}$ consisting of a null sequence of cellular sets, and $M/A(\varepsilon) \not\cong M$.

PROOF. Part of the proof of the forward implication can be found in [**D1**, p. 175]. For completeness, the entire proof is presented. Assume that $M/\tilde{G}$ satisfies DD_3 but not DDP. Then by Corollary 1 and [**Ed**], there exists a decomposition $\tilde{G}$ of M with $\pi_{\tilde{G}}(N_{\tilde{G}})$ of dimension 0, and $M/\tilde{G} \cong M/G \not\cong M$. Let $\{F_i\}$, $1 \leqslant i < \infty$, be a countable dense subset of the space of maps from B^2 into M/G. By an argument similar to that used in the proof of Proposition 2, we can require that, for each $x \in M/G$, $\bigcup_{i=1}^{\infty} F_i^{-1}(x)$ has cardinality less than 3. Let p be the projection map from M onto $M/\tilde{G}$.

Each $M/p^{-1}(F_i(B^2))$ satisfies the DDP by Lemma 1. It follows from [**Ed** and **Ev**, p. 15] that p is approximable by a CE map q from M onto $M/\tilde{G}$, such that q is 1-1 over $\bigcup_{i=1}^{\infty} F_i(B^2)$ and over $M/\tilde{G} - S$ where S is a 0-dimensional set containing $p(N_{\tilde{G}})$. Let G' be the decomposition such that $N_{G'} = N_q$. Then G' satisfies (1).

By [**En**, p. 134], each $F_i(B^2)$ has dimension less than or equal to 3. (In fact, the F_i can be chosen so that each $F_i(B^2)$ has dimension $= 2$. See [**D2**, Theorem 3.3].) Let $F = \bigcup_{i=1}^{\infty} F_i(B^2)$. Then F is an at most 3-dimensional $F\sigma$ set. So the amalgamation lemma can be applied to obtain for each ε, an ε-amalgamation $A(\varepsilon)$ of G', so that $H_{A(\varepsilon)}$ consists of a null sequence, and so that $P(N_{A(\varepsilon)}) \cap F = \varnothing$. Lemma 3 allows one to assume that $M/A(\varepsilon) \not\cong M$. It remains to check that $A(\varepsilon)$ is cellular.

Each point x in $M/A(\varepsilon)$ satisfies McMillan's Cellularity Criterion [**M**, p. 328] since $\pi_{A(\varepsilon)}(N_{A(\varepsilon)}) \cap \pi_{A(\varepsilon)}(F) = \varnothing$. This implies that each element of $A(\varepsilon)$ satisfies McMillan's Cellularity Criterion in M, and so by [**M**, p. 328], $A(\varepsilon)$ is cellular.

For the proof of the reverse implication, assume that (1) and (2) hold. By [**Ed**], G fails to satisfy the DDP. Choose $\varepsilon > 0$ and maps f_e, $1 \leqslant e \leqslant 3$, from B^2 into M/G. Let A be an $\varepsilon/4$ amalgamation of G' as in (2). Let $p = \pi_A \circ \pi_G^{-1}$. For each x in M/G, let $U(x)$ be the $\varepsilon/2$ neighborhood of x. The fact that G and A are uscd, along with the choice of A, yields the result that each $p(U(x))$ contains a nonempty open set $V(x)$.

Cover $\bigcup_{e=1}^{3} f_e(B^2)$ by a finite number of these sets, $V_1, \ldots, V_s$. Let $\delta =$ the Lebesgue number of the cover $\{(V_i)\}$, $1 \leqslant i \leqslant s$, of $\bigcup_{e=1}^{3} f_e(B^2)$. Proposition 1 implies that there exist $\delta/3$ approximations F_e to $p \circ f_e$, $1 \leqslant e \leqslant 3$, so that $\bigcap_{e=1}^{3} F_e(B^2) = \varnothing$. Choose approximate lifts $\tilde{f}_e$: $B^2 \to M/G$ to the F_e so that

$p(p \circ \tilde{f}_e, F_e) < \delta/3$, and so that $\bigcap_{e=1}^{3} p \circ \tilde{f}_e(B^2) = \varnothing$. It follows that the $\tilde{f}_e$ are ε approximations to the f_e, $1 \leqslant e \leqslant 3$, with $\bigcap_{e=1}^{3} \tilde{f}_e(B^2) = \varnothing$. This completes the proof.

REMARK 3. The characterization given in Theorem 2 is nontrivial in the following sense. Not all cellular decompositions of M are intrinsically 0-dimensional [DG] and there exist 0-dimensional cellular decompositions failing to satisfy DD_3 [G2].

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