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SHORTER NOTES

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THE RIEMANN INTEGRAL IN CONSTRUCTIVE MATHEMATICS

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ABSTRACT. A suitable Riemann integral in one variable is developed for use in constructive mathematics.

In this note we briefly define and prove the main properties of the Riemann integral (one variable) in a manner which is suitable for constructive mathematics [1], [2]. In particular we correct the development of the Riemann integral in Bishop [1]. Our definition generalizes easily to multiple Riemann integrals; however, we postpone to a subsequent paper the discussion of multiple integrals where also Stokes' theorem will be proved.

The main difficulty is that the constructive real numbers are not totally ordered. In particular, two partitions of an interval (defined classically) may not have a common refinement.

To begin, we first define $\int_a^b f(t)\,dt$ where f is a real valued uniformly continuous map on the interval $[a, b]$, $a < b$. We consider only partitions of $[a, b]$ of the following form: $P \equiv a_0 = a < a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_n < b = a_{n+1}$, where a_i is rational, $i = 1, 2, 3, \ldots, n$. Note the strict inequality at the "extremities" of P. The point is that two such partitions admit a common refinement (the rationals are totally ordered).

Notation. (a) mesh $P = \sup\{a_{i+1} - a_i \mid i = 0, 1, 2, 3, \ldots, n\}$,

(b) $S(f; P) = \sum_{i=0}^{n} f(a_i)(a_{i+1} - a_i)$.

The standard classical arguments now apply to prove that $\lim_{\text{mesh } P \to 0} S(f; P)$ exists (cf. [1]). Let $\int_a^b f \equiv \int_a^b f(t)\,dt$ denote this limit.

LEMMA 1. *Let f: $[a, b] \to R$ be uniformly continuous, $a < b$. Let $c \in (a, b)$. Then $\int_a^b f = \int_a^c f + \int_c^b f$.*

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PROOF. The standard classical argument applies, subject to the following modification: Let $P \equiv a < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n < c$; $Q \equiv c < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m < b$, be partitions of $[a, c]$, $[c, b]$, respectively. Let d be a rational approximation to c such that $a_n < d < c_1$. Let $R \equiv a < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n < d < c_1 \leqslant c_2 \leqslant \cdots \leqslant c_m < b$. It is clear that for mesh P, mesh Q, $|d - c|$ sufficiently small, $S(f; R)$ is arbitrarily close to $\int_a^b f(t)\,dt$. Thus Lemma 1 is proved. Let f be uniformly continuous on an open interval J. One fixes $a, b \in J$.

We define $\int_a^b f$ in the following way: Let $x \in J$, $x < \inf\{a, b\}$ [recall that J is open].

DEFINITION. $\int_a^b f = \int_x^b f - \int_x^a f$.

LEMMA 2. $\int_a^b f$ is well defined.

PROOF. Let $x, y \in J$; $x < \inf\{a, b\}$; $y < \inf\{a, b\}$. It is required to prove that

$$\int_x^b f - \int_x^a f = \int_y^b f - \int_y^a f.$$

To this end, let $z \in J$, $z < \inf\{x, y\}$. Applying Lemma 1,

$$\int_x^b f = \int_z^b f - \int_z^x f; \qquad \int_x^a f = \int_z^a f - \int_z^x f.$$

Consequently,

$$\int_x^b f - \int_x^a f = \int_z^b f - \int_z^a f.$$

Similarly,

$$\int_y^b f - \int_y^a f = \int_z^b f - \int_z^a f.$$

This completes the proof of Lemma 2.

COROLLARY. *Let* $a, b, c \in J$. *Then* $\int_a^b f = \int_a^c f + \int_c^b f$. [*One chooses* $x \in J$, $x < \inf\{a, b, c\}$.]

REMARK. Evidently, the above corollary permits one to prove the relation $d[\int_a^x f]/dx = f(x)$.

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