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A NOTE ON NOETHERIAN P.I. RINGS

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ABSTRACT. Let $R = F\{x_1, \ldots, x_k\}$ be an affine semiprime p.i. ring, Krull-dim$(R) = 2$ and F a central subfield. It is proved that R is noetherian iff R is a left (right) finite module over a commutative affine subring.

Introduction. In **[C]** and later in **[Sch]** examples were given (for the first time) of prime affine Noetherian p.i. rings which *are not* finite modules over their centers. The common feature that both examples share is that they are finite modules over a commutative Noetherian subring. Also, both examples are of Krull dimension 2. We show here that this is not accidental. Indeed we have the following

THEOREM. *Let* $R = F\{x_1, \ldots, x_k\}$ *be a semiprime affine p.i. ring,* $k \cdot d(R) = 2$. *Then* R *is noetherian iff* R *is a finite left (right) module over a commutative affine (hence noetherian) subring.*

For notations we follow **[P]**. Also by $N_R(I)$ we denote the nilradical of the ideal I of R. $T(R)$ will denote the "trace envelope" of R (e.g., **[B]**), $k \cdot d(R)$ will denote the classical Krull dimension of R, and $h_R(I)$ will denote the height of the ideal I in R.

LEMMA 1. *Let* $R = F\{x_1, \ldots, x_k\}$ *be a semiprime affine p.i. ring with* $k \cdot d(R) = 1$. *Then* R *is a finite module over its center.*

PROOF. Let $P_1, \ldots, P_l$ be the minimal prime ideals in R; we have $\bigcap_{i=1}^{l} P_i = \{0\}$. Let $R_i \equiv R/P_i$ for $i = 1, \ldots, l$. Then R_i is a finite module over its center Z_i **[S, Sch]**. We have that $\bigcap_{j\neq i} P_j + P_i/P_i \equiv \overline{\bigcap_{j\neq i} P_j} \neq 0$ in R_i for $i = 1, \ldots, l$. By Noether's normalization theorem, [N] there exists $\bar{a}_i \in \overline{\bigcap_{j\neq i} P_j} \cap Z_i$ such that Z_i is integral over $F[\bar{a}_i]$ for $i = 1, \ldots, l$ (if $k \cdot d(R_t) = 0$ we take $\bar{a}_t = 1$). Let $\nu\colon R \to R_1 \oplus R_2 \oplus \cdots \oplus R_l$ denote the obvious inclusion with $\nu(x) = \bar{x}_1 \oplus \cdots \oplus \bar{x}_l$ where $\bar{x}_i$ denotes the canonical image of x in R_i. Let $a_i \in \bigcap_{j\neq i} P_j$ be a preimage of $\bar{a}_i$ in R, $i = 1, \ldots, l$; then $\nu(a_i) = \bar{a}_i$, and $\nu(a_i) \in Z_i \subseteq Z(R_1 \oplus \cdots \oplus R_l)$. Thus $F[\nu(a_1), \ldots, \nu(a_l)]$ is a central subring of $R_1 \oplus \cdots \oplus R_l$. Moreover $Z_1 \oplus \cdots \oplus Z_l = Z(R_1 \oplus \cdots \oplus R_l)$ is integral over $F[\nu(a_1), \ldots, \nu(a_l)]$ and consequently $\nu(R) \subseteq R_1 \oplus \cdots \oplus R_l$ is a finite module over $\nu(F[a_1, \ldots, a_l])$. Q.E.D.

Let R be a semiprime affine p.l. ring with minimal prime ideals $P_1, \ldots, P_k$. Assume that p.i.$d(P_j) = n_t$ for $i_{t-1} + 1 \leqslant j \leqslant i_t$, $1 \leqslant t \leqslant l$, and $n = n_1 > n_2 > \cdots > n_l$, $i_l = k$. Let $Q_t = \bigcap_{j=i_{t-1}+1}^{i_t} P_j$, $t = 1, \ldots, l$. By [A] there exists a

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central alternating polynomial α_t of $n_t \times n_t$ matrices with the following properties:

(a) If $\alpha_t = \alpha_t[X_1, \ldots, X_m, \ Y_1, \ldots, Y_s]$ then given $\bar{r} \in R/Q_t$ we have $\alpha_t[b_1, \ldots, b_m, d_1, \ldots, d_s]c_i(\bar{r}) = \alpha_t[f_1, \ldots, f_m, d_1, \ldots, d_s]$ where the b_i's, d_i's and f_i's are in R/Q_t and $c_i(\bar{r})$ is the ith coefficient of the characteristic polynomial of $\bar{r}$.

(b) If $e = \alpha_t[b_1, \ldots, b_m, \ d_1, \ldots, d_s]$ then $(e\bar{r})^k + \gamma_1(e\bar{r})^{k-1} + \cdots + \gamma_k = 0$ where γ_i, $i = 1, \ldots, k$, are evaluations of α_t in R/Q_t.

LEMMA 2. *Let $J_t = \{x \in R \mid x$ is an evaluation of α_t with elements in $\bigcap_{j<t-1} Q_j\}$. Then for $x \in J_t$ we have*

(i) $x \in Z(R)$.

(ii) *if* $r \in R$ *then* $(xr)^m + \gamma_1(xr)^{m-1} + \cdots + \gamma_m = 0$ *where* $\gamma_i \in J_t$, $i = 1, \ldots, m$.

(iii) $J_t \subseteq \bigcap_{j \neq t} Q_j$.

(iv) $J_t \cdot J_s = 0$ *for* $t \neq s$, $t, s = 1, \ldots, l$.

PROOF. (i) $[x, R] \subset Q_t \cap \cdots \cap Q_l$ since $x \equiv 0 \pmod{Q_{t+1} \cap \cdots \cap Q_l}$ (α_t vanishes on R/Q_i, $t + 1 \leq i$) and $x \equiv$ central element $(\mathrm{mod}\ Q_t)$. Also $x \in \bigcap_{j<t-1} Q_j$; thus $[x, R] \subset \bigcap_{j=1}^{l} Q_j = \bigcap_{j=1}^{k} P_j = \{0\}$.

(ii) This is a consequence of properties (a) and (b).

(iii) One merely observes that $x \in \bigcap_{j<t-1} Q_j$ by definition and $x \in \bigcap_{j>t+1} Q_j$ since α_t vanishes on R/Q_t for $j \geq t + 1$.

Finally, (iv) is a trivial consequence of (iii).

We now prove our main result:

THEOREM. *Let $R = F\{x_1, \ldots, x_s\}$ be a semiprime affine p.i. ring R and $k \cdot d(R) = 2$. Then R is Noetherian iff R is a finite left (right) module over a commutative affine subring D.*

PROOF. We resume the notations of the previous paragraph. We have that $R/Q_1 \oplus \cdots \oplus R/Q_l$ is Noetherian since R is and, by **[Sch]**, $T \equiv T(R/Q_1) \oplus \cdots \oplus T(R/Q_l)$ is integral over the former ring. Let ν denote the natural inclusion $\nu\colon R/Q_1 \oplus \cdots \oplus R/Q_l \to T$. We have by the previous remarks that $\nu(\sum_t J_t R)$ is a two-sided ideal in T and

$$N_T\left\{\nu\left(\sum_t J_t R\right)\right\} \cap \nu(R) = N_{\nu(R)}\left\{\nu\left(\sum_t J_t R\right)\right\}.$$

Thus

$$\frac{\nu(R)}{N_{\nu(R)}\{\nu(\sum_t J_t R)\}} \subset \frac{T}{N_T(\nu(\sum_t J_t R))}$$

and the inclusion is integral. Clearly, $h_{Z(T)}(\nu(\sum_t J_t)) \geq 1$. By Noether's normalization theorem we have x, y with $x, y \in Z(T)$, $x \in \sum_t \nu(J_t)Z(T)$, and $Z(T)$ is integral over $F[x, y]$. Thus, since

$$k \cdot d\left(\frac{\nu(R)}{N_{\nu(R)}\{\nu(\sum_t J_t R)\}}\right) = 1,$$

there exists an $a \in \nu(R)$ with $y^m + \gamma_1 y^{m-1} + \cdots + \gamma_n \in N_{\nu(R)}\{\nu(\Sigma_t J_t R)\}$, $\gamma_i \in F[a]$, $i = 1, \ldots, m$. Consequently, by (ii) of the previous lemma there are $\beta_1, \ldots, \beta_s$ in $\nu(\Sigma_t J_t)$ such that y is integral over $F[\beta_1, \ldots, \beta_s, a]$. Thus $k[x, y]$ is integral over $F[x, \beta_1, \ldots, \beta_s, a] \equiv D \subset \nu(R)$ and D is commutative. Consequently T is a right and left module over D and $\nu(R)$ therefore shares the same property. The other direction is definitely trivial. Q.E.D.

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