

ON THE ABSOLUTE CONVERGENCE OF LACUNARY FOURIER SERIES

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ABSTRACT. In 1958, P. B. Kennedy [2] proved a two-part theorem on the order of magnitude of the coefficients and the absolute convergence of lacunary Fourier series, and he conjectured that his conclusions remained valid under weaker conditions. The conjecture on the order of magnitude of the coefficients was established by M. and S. I. Izumi [1]. In this note the second half of the conjecture, concerning absolute convergence, is deduced from a recent result of Patadia [3].

1. Introduction. Let

$$(1.1) \quad \sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$$

be the Fourier series of a 2π -periodic function $f \in L[-\pi, \pi]$ with an infinity of gaps (n_k, n_{k+1}) , where $\{n_k\}$ ($k \in \mathbb{N}$) is a strictly increasing sequence of natural numbers. Kennedy has proved the following theorem.

THEOREM [2, THEOREM II]. *If $f \in \text{Lip } \alpha$ ($0 < \alpha < 1$) in a subset E of $[-\pi, \pi]$ of positive spread and if $\{n_k\}$ satisfies the gap condition*

$$(1.2) \quad \frac{n_{k+1} - n_k}{n_k^\beta \log n_k} \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

where $0 < \beta < 1$, then

- (i) $a_n, b_n = O(n^{-\alpha\beta})$, and
- (ii) the Fourier series (1.1) of f converges absolutely when $\alpha > \frac{1}{2}(\beta^{-1} - 1)$.

Kennedy then conjectures that the conclusions of the theorem remain true if the factor $\log n_k$ in (1.2) is suppressed. The conjecture relating to part (i) of the theorem was proved in an even stronger form (involving the replacement of E by a single point) by M. and S. I. Izumi [1, Theorem 1]. The object of this note is to prove a similarly strengthened form of Kennedy's conjecture for part (ii) of his theorem.

We consider the gap condition

$$(1.3) \quad (n_{k+1} - n_k) > C n_k^\beta k^\theta \quad (0 < \beta < 1, \theta \geq 0)$$

where C is a positive constant, and we prove the following theorem.

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THEOREM. If $f \in \text{Lip } \alpha$ ($0 < \alpha < 1$) at some point $x_0 \in (-\pi, \pi)$ and if $\{n_k\}$ satisfies (1.3), then

$$\sum_{k=1}^{\infty} (|a_{n_k}|^r + |b_{n_k}|^r) < \infty \quad (0 < r \leq 1),$$

when $\alpha\beta r + \alpha\theta > (1 - r/2)(1 - \beta)$.

REMARK. The particular case $r = 1$, $\theta = 0$ of the theorem provides verification of Kennedy's conjecture on the absolute convergence of the Fourier series (1.1), with the additional feature that the set of positive spread is replaced by a single point. The theorem also generalizes Theorem 2 in the paper [1] of M. and S. I. Izumi.

2. Proof of the theorem. We need the following lemma.

LEMMA. Suppose that $\{n_k\}$ satisfies (1.3) and that δ is such that $0 < \delta < (1 + \theta)/(1 - \beta)$. Then there exists an integer k_0 and a real number B such that $0 < B < 1$ and $n_k > Bk^\delta$ for all $k \geq k_0$.

PROOF. Since $\delta < (1 + \theta)/(1 - \beta)$ we get $\delta\beta + \theta > \delta - 1$. Put $\delta\beta + \theta - \delta + 1 = r$, so that $r > 0$. Let M be a positive integer such that $M > (1 + \theta)/(1 - \beta)$, and k_0 be a positive integer such that $Ck_0' > 2^M$. We shall prove the lemma by induction.

Choose B , $0 < B < 1$, such that $n_{k_0} > Bk_0^\delta$.

Assume that p is an integer such that $p \geq k_0$ and $n_p > Bp^\delta$. Then, observing that $B < B^\beta$, we get by (1.3)

$$\begin{aligned} (2.1) \quad n_{p+1} &> n_p + Cn_p^\beta p^\theta > Bp^\delta + CB^\beta p^{\delta\beta+\theta} \\ &> Bp^\delta + CBp^{\delta-1+r} = Bp^\delta(1 + Cp^r/p). \end{aligned}$$

But

$$\begin{aligned} Cp^r &\geq Ck_0' > 2^M - 1 = \binom{M}{1} + \binom{M}{2} + \binom{M}{3} + \cdots + \binom{M}{M} \\ &> \binom{M}{1} + \binom{M}{2} \frac{1}{p} + \binom{M}{3} \frac{1}{p^2} + \cdots + \binom{M}{M} \frac{1}{p^{M-1}}, \end{aligned}$$

and therefore

$$\begin{aligned} (2.2) \quad 1 + \frac{Cp^r}{p} &> 1 + \frac{1}{p} \left[\binom{M}{1} + \binom{M}{2} \frac{1}{p} + \binom{M}{3} \frac{1}{p^2} + \cdots + \binom{M}{M} \frac{1}{p^{M-1}} \right] \\ &= (1 + 1/p)^M > (1 + 1/p)^\delta. \end{aligned}$$

Hence, by (2.1) and (2.2)

$$n_{p+1} > Bp^\delta(1 + 1/p)^\delta = B(p+1)^\delta.$$

It then follows by induction that $n_k > Bk^\delta$ for all $k \geq k_0$.

PROOF OF THE THEOREM. We have $\alpha\beta r + \alpha\theta > (1 - r/2)(1 - \beta)$. Hence

$$(1 + \theta)/(1 - \beta) > (2 - r - 2\alpha\theta)/2\alpha\beta r.$$

Choose δ such that

$$(2.3) \quad (1 + \theta)/(1 - \beta) > \delta > (2 - r - 2\alpha\theta)/2\alpha\beta r.$$

Now,

(i) by the lemma $n_k > Bk^\delta$ for all sufficiently large k ;

(ii) since $f \in \text{Lip } \alpha$ at x_0 , $\omega(\delta, f, x_0) = O(\delta^\alpha)$, where

$$\omega(\delta, f, x_0) = \sup_{0 < h < \delta} \{|f(x_0 \pm h) - f(x_0)|\};$$

(iii) by (2.3), $\alpha\beta r\delta + \alpha r\theta + r/2 > 1$.

Therefore we have

$$\sum_{k=1}^{\infty} \frac{(\omega(A/(n_k^\beta k^\theta), f, x_0))^r}{k^{r/2}} = \sum_{k=1}^{\infty} \frac{O(1)}{(n_k^\beta k^\theta)^{\alpha r} k^{r/2}} = \sum_{k=1}^{\infty} \frac{O(1)}{k^{\alpha\beta r\delta + \alpha r\theta + r/2}}$$

and the last series converges. The theorem now follows from [3, Theorem 1].

REFERENCES

1. M. Izumi and S. I. Izumi, *On lacunary Fourier series*, Proc. Japan Acad. Ser. A Math. Sci. **41** (1965), 648–651.
2. P. B. Kennedy, *On the coefficients in certain Fourier series*, J. London Math. Soc. (2) **33** (1958), 196–207.
3. J. R. Patadia, *On the absolute convergence of lacunary Fourier series*, Proc. Amer. Math. Soc. **71** (1978), 19–25.

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