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CONTINUITY OF BEST APPROXIMANTS

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ABSTRACT. Let C_n, $n \in \mathbf{N}$, be Φ-closed lattices in an Orlicz-space $L_\Phi(\Omega, \mathcal{A}, \mu)$ and assume that C_n increases or decreases to a Φ-closed lattice C_∞. Let f_n, $n \in \mathbf{N}$, be $\mathcal{A}$-measurable real valued functions with $f_n \to f$ μ-a.e. and $\sup|f_n| \in L_\Phi$. If g_n is a best Φ-approximant of f_n in C_n it is shown that $\underline{\lim}_{n\in\mathbf{N}}\, g_n$ and $\overline{\lim}_{n\in\mathbf{N}}\, g_n$ are best Φ-approximants of f in C_∞.

1. Introduction and notations. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\Phi: \mathbf{R}_+ \to \mathbf{R}_+$ be a convex function with $\Phi(0) = 0$ and $\Phi \not\equiv 0$. Denote by $L_\Phi(\Omega, \mathcal{A}, \mu)$ respectively $L_\Phi^\infty(\Omega, \mathcal{A}, \mu)$ the system of all μ-equivalence classes of $\mathcal{A}$-measurable functions f such that $\int \Phi(\alpha|f|)\, d\mu < \infty$ for some $\alpha > 0$ respectively for all $\alpha > 0$. L_Φ and L_Φ^∞ are linear spaces with $L_\Phi^\infty \subset L_\Phi$; if $\Phi(x) = x^p$ then $L_\Phi = L_\Phi^\infty$ and we obtain the spaces L_p, $p \geqslant 1$. If $C \subset L_\Phi$ and $f \in L_\Phi$ denote by $\mu_\Phi(f|C)$ the system of all $g \in C$ fulfilling

$$\int \Phi(|f - g|)\, d\mu = \inf_{h \in C} \int \Phi(|f - h|)\, d\mu.$$

The elements of $\mu_\Phi(f|C)$ are called *best Φ-approximants* of f, given C. The concept of best Φ-approximants, given C, covers and unifies many important concepts of probability theory, e.g. the concepts in **[1]**, **[2]**, **[6]**; for more details see **[4]**. It is known that $\mu_\Phi(f|C) \neq \varnothing$ if C is a lattice (i.e. $f, g \in C$ implies $f \wedge g, f \vee g \in C$) which is *$\Phi$-closed* (i.e. $f_n \in C, f \in L_\Phi$ and $f_n \uparrow f$ or $f_n \downarrow f$ imply $f \in C$); see Theorem 4 of **[4]**. In general, $\mu_\Phi(f|C)$ contains a lot of different elements; for instance if $\Phi(x) = x$ or if C is not convex. This creates problems for proving limit theorems for best Φ-approximants. In special cases–i.e. for $\Phi(x) = x^p$, $p > 1$, and special types of C–limit results for best Φ-approximants of f, given C, are easier to obtain for varying f than for varying C; but in all these cases best approximants are unique. In the general context, however, the case of varying f is more complex. There exist limit theorems for best Φ-approximants of martingale type (see Theorem 21 and Theorem 22 of **[4]**)–i.e. limit theorems for $g_n \in \mu_\Phi(f|C_n)$ with varying C_n–but there exist no continuity theorems for best Φ-approximants–i.e. limit theorems for $g_n \in \mu_\Phi(f_n|C)$ with varying f_n. It is the aim of this paper to close this gap. We prove a limit theorem for best Φ-approximants $g_n \in \mu_\Phi(f_n|C_n)$ where as well the functions f_n as the Φ-closed lattices C_n may vary with $n \in \mathbf{N}$. We apply this result to obtain continuity of best approximants in the Orlicz-space norm of L_Φ.

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2. The results. Throughout the following let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\Phi: \mathbf{R}_+ \to \mathbf{R}_+$ be a convex function with $\Phi(0) = 0$ and $\Phi \not\equiv 0$. Then Φ is a continuous function with $\lim_{t \nearrow \infty} \Phi(t) = \infty$. If $C_n \subset L_\Phi$, $n \in \mathbf{N} \cup \{\infty\}$, we write $C_n \downarrow C_\infty$ if $C_n \supset C_{n+1}$, $n \in \mathbf{N}$, and $C_\infty = \bigcap_{n \in \mathbf{N}} C_n$. If C_n are Φ-closed lattices we write $C_n \uparrow C_\infty$ if $C_n \subset C_{n+1}$ and C_∞ is the smallest Φ-closed set containing $\bigcup_{n \in \mathbf{N}} C_n$; then C_∞ is a lattice, too (see [**4**, p. 229]).

1. THEOREM. *Assume that* $L_\Phi = L_\Phi^\infty$. *Let* $C_n \subset L_\Phi$, $n \in \mathbf{N}$, *be* Φ*-closed lattices with* $C_n \downarrow C_\infty$ *or* $C_n \uparrow C_\infty$ *and* $f_n \in L_\Phi$, $n \in \mathbf{N}$, *with* $f_n \to f$ μ*-a.e. and* $\sup_{n \in \mathbf{N}} |f_n| \in L_\Phi$. *Then for all* $g_n \in \mu_\Phi(f_n | C_n)$, $n \in \mathbf{N}$,

(i) $\underline{\lim}_{n \in \mathbf{N}}\, g_n \in \mu_\Phi(f | C_\infty)$, $\overline{\lim}_{n \in \mathbf{N}}\, g_n \in \mu_\Phi(f | C_\infty)$.

(ii) $\sup_{n \in \mathbf{N}} |g_n| \in L_\Phi$.

PROOF. Let $C_n \downarrow C_\infty$. We prove that for each $g \in \mu_\Phi(f | C_\infty)$

$$g \wedge \underline{\lim}_{n \in \mathbf{N}}\, g_n \in \mu_\Phi(f | C_\infty), \qquad g \vee \overline{\lim}_{n \in \mathbf{N}}\, g_n \in \mu_\Phi(f | C_\infty). \tag{1}$$

As

$$\underline{\lim}_{n \in \mathbf{N}}\, g_n = \Big(g \vee \overline{\lim}_{n \in \mathbf{N}}\, g_n\Big) \wedge \underline{\lim}_{n \in \mathbf{N}}\, g_n$$

and

$$\overline{\lim}_{n \in \mathbf{N}}\, g_n = \Big(g \wedge \underline{\lim}_{n \in \mathbf{N}}\, g_n\Big) \vee \overline{\lim}_{n \in \mathbf{N}}\, g_n,$$

(1) implies (i).

Applying Lemma 3 to $C_k \supset \cdots \supset C_n \supset C_\infty$ we obtain for $n > k$

$$\begin{aligned} &\int \Phi(|f_k \wedge \cdots \wedge f_n \wedge f - g_k \wedge \cdots \wedge g_n \wedge g|)\, d\mu \\ &\qquad < \int \Phi(|f_k \wedge \cdots \wedge f_n \wedge f - g|)\, d\mu \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\int \Phi(|f_k \vee \cdots \vee f_n \vee f - g_k \vee \cdots \vee g_n \vee g|)\, d\mu \\ &\qquad < \int \Phi(|f_k \vee \cdots \vee f_n \vee f - g|)\, d\mu. \end{aligned} \tag{3}$$

From (2) we obtain for each k with $n \to \infty$ according to the Lemma of Fatou that

$$\begin{aligned} &\int \Phi\Big(\Big|f \wedge \bigwedge_{n > k} f_n - g \wedge \bigwedge_{n > k} g_n\Big|\Big)\, d\mu \\ &\qquad < \underline{\lim}_{n \in \mathbf{N}} \int \Phi(|f_k \wedge \cdots \wedge f_n \wedge f - g|)\, d\mu. \end{aligned} \tag{4}$$

Since $\sup_{n \in \mathbf{N}} |f_n| \in L_\Phi$ by assumption, (4) implies by the Theorem of Lebesgue that for each $k \in \mathbf{N}$

$$\int \Phi\Big(\Big|f \wedge \bigwedge_{n > k} f_n - g \wedge \bigwedge_{n > k} g_n\Big|\Big)\, d\mu \leqslant \int \Phi\Big(\Big|f \wedge \bigwedge_{n > k} f_n - g\Big|\Big)\, d\mu < \infty. \tag{5}$$

As $f_n \to f$, $\sup_{n\in\mathbf{N}}|f_n| \in L_\Phi$ we obtain from (5) with $k \to \infty$ using on the left side the Lemma of Fatou and on the right side the Theorem of Lebesgue that

$$(6) \qquad \int \Phi\left(\left|f - g \wedge \varliminf_{n\in\mathbf{N}} g_n\right|\right) d\mu \leqslant \int \Phi(|f - g|)\, d\mu < \infty.$$

From (5) we obtain that $g \wedge \bigwedge_{n\geqslant k} g_n \in L_\Phi$. As $g \wedge \bigwedge_{n\geqslant k} g_n \leqslant \bigwedge_{n\geqslant k} g_n \leqslant g_k$ this implies

$$(7) \qquad \bigwedge_{n\geqslant k} g_n \in L_\Phi.$$

In the same way as (6) and (7) we obtain

$$(6)^* \qquad \int \Phi\left(\left|f - g \vee \varlimsup_{n\in\mathbf{N}} g_n\right|\right) d\mu \leqslant \int \Phi(|f - g|)\, d\mu < \infty$$

and

$$(7)^* \qquad \bigvee_{n\geqslant k} g_n \in L_\Phi.$$

From (7) and (7)* applied to $k = 1$ we obtain (ii). As $C_n\downarrow$ are Φ-closed lattices we obtain with (7) that $\bigwedge_{n\geqslant j} g_n \in C_k$ for $j \geqslant k$. As $\varliminf_{n\in\mathbf{N}} g_n \in L_\Phi$ by (ii), this implies $\varliminf_{n\in\mathbf{N}} g_n \in C_k$ for each $k \in \mathbf{N}$ and hence $\varliminf_{n\in\mathbf{N}} g_n \in C_\infty$. Similarly $\varlimsup_{n\in\mathbf{N}} g_n \in C_\infty$. Now (6), (6)* and $g \in C_\infty$ imply (1). This finishes the proof for the decreasing case.

Now let $C_n\uparrow C_\infty$. Applying Lemma 3 with $C_n \supset C_{n-1} \supset \cdots \supset C_k$, $k < n$, we obtain for $k < n$

$$\int \Phi(|f_k \wedge \cdots \wedge f_n - g_k \wedge \cdots \wedge g_n|)\, d\mu \leqslant \int \Phi(|f_k \wedge \cdots \wedge f_n - g_k|)\, d\mu.$$

Proceeding now as in the decreasing case, i.e. letting at first $n \to \infty$ and then $k \to \infty$ and using on the left sides the Lemma of Fatou and on the right sides the Theorem of Lebesgue we obtain

$$(8) \qquad \bigwedge_{n\geqslant k} g_n \in L_\Phi, \qquad k \in \mathbf{N},$$

and

$$(9) \qquad \int \Phi\left(\left|f - \varliminf_{n\in\mathbf{N}} g_n\right|\right) d\mu \leqslant \varliminf_{k\in\mathbf{N}} \int \Phi\left(\left|\bigwedge_{n\geqslant k} f_n - g_k\right|\right) d\mu.$$

In the same way we obtain

$$(8)^* \qquad \bigvee_{n\geqslant k} g_n \in L_\Phi, \qquad k \in \mathbf{N},$$

and

$$(9)^* \qquad \int \Phi\left(\left|f - \varlimsup_{n\in\mathbf{N}} g_n\right|\right) d\mu \leqslant \varliminf_{k\in\mathbf{N}} \int \Phi\left(\left|\bigvee_{n\geqslant k} f_n - g_k\right|\right) d\mu < \infty.$$

Relations (8) and (8*) directly imply

$$(10) \qquad \sup_{n\in\mathbf{N}} |g_n| \in L_\Phi \quad \text{and} \quad \varliminf_{n\in\mathbf{N}} g_n,\ \varlimsup_{n\in\mathbf{N}} g_n \in C_\infty.$$

Now apply Lemma 4 to $h_k = |f_k - g_k|$ and $r_k := |\bigwedge_{n>k} f_n - f_k|$. Since $\sup_{k\in\mathbf{N}}|g_k| \in L_\Phi$ by (10), $\sup_{k\in\mathbf{N}}|f_k| \in L_\Phi$ and $f_k \to f$ μ-a.e. by assumption we have $\sup_{k\in\mathbf{N}} h_k$, $\sup_{k\in\mathbf{N}} r_k \in L_\Phi$ and $r_k \to 0$ μ-a.e., i.e. the assumptions of Lemma 4 are fulfilled. Hence we obtain

$$\text{(11)} \qquad \varliminf_{k\in\mathbf{N}} \int \Phi(h_k + r_k)\, d\mu = \varliminf_{k\in\mathbf{N}} \int \Phi(h_k)\, d\mu.$$

Since

$$\Phi\left(\left|\bigwedge_{n>k} f_n - g_k\right|\right) \leq \Phi\left(|f_k - g_k| + \left|\bigwedge_{n>k} f_n - f_k\right|\right) = \Phi(h_k + r_k),$$

(9) and (11) imply

$$\text{(12)} \qquad \int \Phi\left(\left|f - \varliminf_{n\in\mathbf{N}} g_n\right|\right) d\mu \leq \varliminf_{k\in\mathbf{N}} \int \Phi(|f_k - g_k|)\, d\mu.$$

According to (12) and (10) we get $\varliminf_{n\in\mathbf{N}} g_n \in \mu_\Phi(f|C_\infty)$ if we show that for all $g \in C_\infty$

$$\text{(13)} \qquad \varliminf_{k\in\mathbf{N}} \int \Phi(|f_k - g_k|)\, d\mu \leq \int \Phi(|f - g|)\, d\mu.$$

Let $\hat{C}$ be the set of all $g \in L_\Phi$ fulfilling (13). Since $g_k \in \mu_\Phi(f_k|C_k)$, $C_k\uparrow$, $f_k \to f$ μ-a.e. and $\sup_{n\in\mathbf{N}}|f_n| \in L_\Phi$ it is easy to see that $\hat{C}$ is Φ-closed with $\bigcup_{n\in\mathbf{N}} C_n \subset \hat{C}$. Hence $C_\infty \subset \hat{C}$, i.e. (13) holds for all $g \in \hat{C}$. Thus $\varliminf_{n\in\mathbf{N}} g_n \in \mu_\Phi(f|C_\infty)$ is shown; the proof for $\varlimsup_{n\in\mathbf{N}} g_n \in \mu_\Phi(f|C_\infty)$ runs similarly (by using (9*) instead of (9)).

The martingale results, given in [4], hold for more general functions Φ than convex functions, namely for so-called μ-functions. We do not know whether also the preceding theorem is true for this more general concept; the proof of Theorem 1 heavily uses the convexity of Φ. Approximating $\varliminf_{n\in\mathbf{N}} g_n$ μ-a.e. by g_{τ_n} where τ_n is a sequence of finite stopping times for g_n, $n \in \mathbf{N}$, it can be seen that Theorem 1 is true for μ-functions in the special case that C_n is the system of $\mathcal{C}_n$-measurable functions in L_Φ, where $\mathcal{C}_n \subset \mathcal{C}$ are σ-fields, and $\mathcal{C}_n \uparrow \mathcal{C}_\infty$ or $\mathcal{C}_n \downarrow \mathcal{C}_\infty$. However, this procedure fails for arbitrary Φ-closed lattices C_n.

If $f \in L_\Phi$ put $\|f\|_\Phi := \inf\{a > 0: \int \Phi(|f|/a)\, d\mu \leq 1\}$. Then $\| \ \|_\Phi$ is a norm on L_Φ and the spaces $(L_\Phi, \| \ \|_\Phi)$ are Banach-spaces; the well-known Orlicz spaces (see [5, p. 46]). If $C \subset L_\Phi$ and $f \in L_\Phi$ we denote by $\mu_{\| \ \|}(f|C)$ the set of all *best* $\| \ \|_\Phi$-*approximants* of f, given C, i.e. the set of all elements $g \in C$ with

$$\|f - g\|_\Phi = \inf\{\|f - h\|_\Phi : h \in C\}.$$

The concept of best $\| \ \|_\Phi$-approximants and its connection with the concept of best Φ-approximants has been investigated in [4]. If Φ is strictly convex and if $L_\Phi = L_\Phi^\infty$, then for each Φ-closed convex lattice $C \subset L_\Phi$ and each $f \in L_\Phi$ there exist a unique best Φ-approximant and a unique best $\| \ \|_\Phi$-approximant of f, given C (see Corollary 5 and Corollary 13 of [4]), we denote these unique elements by $\mu_\Phi(f|C)$

and $\mu_{\| \ \|}(f|C)$, respectively. Hence $L_\Phi \ni f \to \mu_\Phi(f|C) \in L_\Phi$ and $L_\Phi \ni f \to \mu_{\| \ \|}(f|C) \in L_\Phi$ are operators on L_Φ and the following result states the $\| \ \|_\Phi$-continuity of these operators.

2. Corollary. *Let Φ be strictly convex and assume that $L_\Phi = L_\Phi^\infty$. Let $C \subset L_\Phi$ be a Φ-closed convex lattice and a cone. Then $\mu_\Phi(\cdot|C)$ and $\mu_{\| \ \|}(\cdot|C)$ are $\| \ \|_\Phi$-continuous operators on L_Φ.*

Proof. As $L_\Phi = L_\Phi^\infty$ let us at first remark that

(1) $$\|h_n\|_\Phi \to 0 \text{ iff } \int \Phi(a|h_n|)\, d\mu \to 0 \text{ for all } a > 0.$$

Let now $\|f_n - f_0\|_\Phi \to_{n\in\mathbf{N}} 0$ and $\mathbf{N}_1 \subset \mathbf{N}$ be a subsequence. It suffices to prove that there exists a subsequence $\mathbf{N}_2 \subset \mathbf{N}_1$ such that

(2) $$\| \mu_\Phi(f_n|C) - \mu_\Phi(f_0|C)\|_\Phi \underset{n\in\mathbf{N}_2}{\to} 0,$$

(3) $$\| \mu_{\| \ \|}(f_n|C) - \mu_{\| \ \|}(f_0|C)\|_\Phi \underset{n\in\mathbf{N}_2}{\to} 0.$$

Since $\|f_n - f_0\|_\Phi \to_{n\in\mathbf{N}_1} 0$ there exists a subsequence $\mathbf{N}_2 \subset \mathbf{N}_1$ such that

(4) $$f_n \underset{n\in\mathbf{N}_2}{\to} f_0 \quad \mu\text{-a.e.}$$

and

(5) $$\sum_{n\in\mathbf{N}_2} \|f_n - f_0\|_\Phi < \infty.$$

From (5) and $L_\Phi = L_\Phi^\infty$ we obtain

(6) $$\sup_{n\in\mathbf{N}_2} |f_n| < |f_0| + \sum_{n\in\mathbf{N}_2} |f_n - f_0| \in L_\Phi.$$

Now (4), (6) and Theorem 1 imply

(7) $$\mu_\Phi(f_n|C) \underset{n\in\mathbf{N}_2}{\to} \mu_\Phi(f_0|C) \quad \mu\text{-a.e.}; \qquad \sup_{n\in\mathbf{N}_2} |\mu_\Phi(f_n|C)| \in L_\Phi.$$

Using (1), $L_\Phi = L_\Phi^\infty$ and the Theorem of Lebesgue, (7) implies (2). It remains to prove (3). Since $\|f_n - f_0\|_\Phi \to 0$ and C is $\| \ \|_\Phi$-closed (see Theorem 10 of [4]) it is easy to see that

(8) $$\delta_n := \|f_n - \mu_{\| \ \|}(f_n|C)\|_\Phi \underset{n\in\mathbf{N}}{\to} \|f_0 - \mu_{\| \ \|}(f_0|C)\|_\Phi =: \delta_0.$$

Let w.l.g. $\delta_0 > 0$; hence w.l.g. $\delta_n > 0$ for all $n \in \mathbf{N}$. According to Corollary 8 of [4] we have, as C is a cone, that

(9) $$\mu_{\| \ \|}(f_n|C) = \delta_n \mu_\Phi\left(\frac{1}{\delta_n} f_n \Big| C\right), \qquad n \in \mathbf{N} \cup \{0\}.$$

Since $f_n \to_{\| \ \|_\Phi} f_0$, and $\delta_n \to \delta_0$ by (8), we have

$$\frac{1}{\delta_n} f_n \underset{\| \ \|_\Phi}{\to} \frac{1}{\delta_0} f_0.$$

Hence the continuity of $\mu_\Phi(\cdot|C)$ implies

$$\left\|\mu_\Phi\left(\frac{1}{\delta_n}f_n|C\right) - \mu_\Phi\left(\frac{1}{\delta_0}f_0|C\right)\right\|_\Phi \underset{n\in\mathbf{N}}{\to} 0.$$

Together with (9) and (8) this yields (3).

For the special case that C is the system of measurable functions with respect to a σ-field the assertion of Corollary 2 follows from Satz 5.10 of [3]. The methods used there are closely related to this special type of C and cannot be transferred to arbitrary Φ-closed convex lattices.

The following lemmas are the main tools for the proof of Theorem 1.

3. Lemma. *Assume that $L_\Phi = L_\Phi^\infty$. Let $C_i \subset L_\Phi$, $i = 1, \ldots, n$, be Φ-closed lattices with $C_1 \supset C_2 \supset \cdots \supset C_n$. If $f_i \in L_\Phi$ and $g_i \in \mu_\Phi(f_i|C_i)$, $i = 1, \ldots, n$ then*
(i) $\int \Phi(|f_1 \wedge \cdots \wedge f_n - g_1 \wedge \cdots \wedge g_n|)\, d\mu \leqslant \int \Phi(|f_1 \wedge \cdots \wedge f_n - g_n|)\, d\mu$,
(ii) $\int \Phi(|f_1 \vee \cdots \vee f_n - g_1 \vee \cdots \vee g_n|)\, d\mu \leqslant \int \Phi(|f_1 \vee \cdots \vee f_n - g_n|)\, d\mu$.

Proof. To show (i) it suffices to prove that for $j < n$

$$\begin{aligned}\int \Phi(|f_1 \wedge \cdots \wedge f_n - g_j \wedge \cdots \wedge g_n|)\, d\mu \\ \leqslant \int \Phi(|f_1 \wedge \cdots \wedge f_n - g_{j+1} \wedge \cdots \wedge g_n|)\, d\mu.\end{aligned} \tag{1}$$

As Φ is convex, Lemma 20 of [4] implies

$$\begin{aligned}\Phi(|f_1 \wedge \cdots \wedge f_n - g_j \wedge (g_{j+1} \wedge \cdots \wedge g_n)|) \\ +\Phi(|f_j - g_j \vee (g_{j+1} \wedge \cdots \wedge g_n)|) \\ \leqslant \Phi(|f_1 \wedge \cdots \wedge f_n - g_{j+1} \wedge \cdots \wedge g_n|) + \Phi(|f_j - g_j|).\end{aligned} \tag{2}$$

Since C_j is a lattice and $g_i \in C_i \subset C_j$ for $i \geqslant j$ we have $g_j \vee (g_{j+1} \wedge \cdots \wedge g_n) \in C_j$. As $g_j \in \mu_\Phi(f_j|C_j)$ we obtain

$$\int \Phi(|f_j - g_j|)\, d\mu \leqslant \int \Phi(|f_j - g_j \vee (g_{j+1} \wedge \cdots \wedge g_n)|)\, d\mu. \tag{3}$$

Using (3) integration of (2) yields (1). This proves (i); the proof for (ii) runs by interchanging $\vee$ and $\wedge$.

4. Lemma. *Assume that $L_\Phi = L_\Phi^\infty$. Let $0 \leqslant h_k$, $r_k \in L_\Phi$ and assume that $\sup_{k\in\mathbf{N}} h_k$, $\sup_{k\in\mathbf{N}} r_k \in L_\Phi$ and $r_k \to 0$ μ-a.e. Then*

$$\int \Phi(h_k + r_k)\, d\mu - \int \Phi(h_k)\, d\mu \underset{k\in\mathbf{N}}{\to} 0.$$

Proof. Let Φ'_+ be the right derivative of Φ. Then Φ'_+ is nondecreasing and $\Phi(x) = \int_0^x \Phi'_+(t)\, dt$ (see e.g. [5]). Hence for all $k \in \mathbf{N}$

$$(*)\quad \Phi(h_k + r_k) - \Phi(h_k) = \int_{h_k}^{h_k + r_k} \Phi'_+(t)\, dt \leqslant r_k \Phi'_+(h_k + r_k) \leqslant r\Phi'_+(h + r)$$

with $r := \sup_{k\in\mathbf{N}} r_k \in L_\Phi$ and $h := \sup_{k\in\mathbf{N}} h_k \in L_\Phi$.

By the Theorem of Lebesgue $(*)$ directly implies the assertion if we show $r\Phi'_+(h+r) \in L_1$. As $0 \leqslant x\Phi'_+(x) \leqslant \int_x^{2x} \Phi'_+(t)\,dt \leqslant \Phi(2x)$ and $L_\Phi = L_\Phi^\infty$ we have $g\Phi'_+(g) \in L_1$ if $0 \leqslant g \in L_\Phi$. Applying this to $g = h + r \in L_\Phi$ we obtain $r\Phi'_+(h+r) \in L_1$.

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