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CONVERGENCE OF BEST BEST L_∞-APPROXIMATIONS

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ABSTRACT. Let $(\Omega, \mathcal{Q}, \mu)$ be a probability space and let $\{\mathfrak{B}_i\}_{i=1}^\infty$ be an increasing sequence of subsigma algebras of $\mathcal{Q}$. Let $A = L_\infty(\Omega, \mathcal{Q}, \mu)$, let $B_i = L_\infty(\Omega, \mathfrak{B}_i, \mu)$, $i \geq 1$, and let $f \in A$. Let f_i denote the best best L_∞-approximation to f by elements of B_i. It is shown that $\lim_i f_i(x)$ exists a.e.

We begin with a brief introduction in which we present notation and terminology, a related result [1] for $1 < p < \infty$, and some results from [2] that will be used to establish a.e. convergence of the sequence $\{f_i\}$.

Let $\mathfrak{B}_\infty$ denote the subsigma algebra generated by the algebra $\bigcup_i \mathfrak{B}_i$. For $1 < p < \infty$, $1 \leqslant i \leqslant \infty$ and $g \in L_p(\Omega, \mathcal{Q}, \mu)$, let $g_{i,p}$ denote the best L_p-approximation to g by elements of $L_p(\Omega, \mathfrak{B}_i, \mu)$; T. Ando and I. Amemiya [1] showed that $\lim_i g_{i,p} = g_{\infty,p}$ a.e. and in L_p. It is shown in [2] that if $g \in A$, then for $1 \leqslant i \leqslant \infty$, $\lim_p g_{i,p}$ exists a.e. and is the best best L_∞-approximation, $g_{i,\infty}$, to g by $\mathfrak{B}_i$-measurable functions. To simplify the notation, we let f be a fixed element of A and, without loss of generality, we suppose that $0 \leqslant f \leqslant 1$; furthermore, we denote $f_{i,\infty}$ by f_i. Our proof of the fact that the sequence $\{f_i\}$ converges a.e. uses some technical results from [2] that we introduce next. To simplify the notation during this introduction, suppress i from $\mathfrak{B}_i$: $\mathfrak{B}$ is a subsigma algebra of $\mathcal{Q}$ and $B = L_\infty(\Omega, \mathfrak{B}, \mu)$.

Let $\mathcal{P}$ denote the set of denumerable partitions of Ω by elements of $\mathfrak{B}$.

For $E \in \mathcal{Q}$, let $O(E)$ denote the essential oscillation of f on E: $O(E) = O(f, E) = \operatorname{essup}(f, E) - \operatorname{essinf}(f, E)$, where $\operatorname{essup}(f, E) = \operatorname{essinf}(f, E) = 0$ if $\mu(E) = 0$ and for $\mu(E) > 0$

$$u(E) = \operatorname{essup}(f, E) = \inf\{\lambda; \mu(\{x \in E; f(x) > \mu\}) = 0\}$$

and

$$l(E) = \operatorname{essinf}(f, E) = \sup\{\lambda; \mu(\{x \in E; f(x) < \lambda\}) = 0\}.$$

Let $d(g, B)$ denote the distance from an element g of A to the subspace B of A.

Next we recall two lemmas from [2]. Lemma 1 shows that $\mathcal{P}$ can be used to estimate $d(f, B)$. Lemma 2 asserts that the flexibility afforded by $\mathcal{P}$ permits us to replace an inf by a min. The partitions corresponding to each min provide an equivalence class of elements of $\mathfrak{B}$. These equivalence classes comprise a monotone family parameterized by the positive reals.

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LEMMA 1. $d = d(f, B) = (1/2) \inf_{\pi \in \mathcal{P}} \sup\{O(E); E \in \pi, \mu(E) > 0\}$.

LEMMA 2. *For $h > 0$ and $\pi \in \mathcal{P}$, let $\delta(h, \pi) = \{\Sigma \mu(E); E \in \pi, O(E) \geqslant h\}$ and let $\delta_h = \inf\{\delta(h, \pi); \pi \in \mathcal{P}\}$. Then there exists π such that $\delta_h = \delta(h, \pi) = \mu(E_\pi^h)$, where $E_\pi^h = \bigcup \{E; E \in \pi, O(E) \geqslant h\}$.*

Lemmas 1 and 2 assure (i) $\delta_h = 0$ if $h > 2d$ and (ii) if $h < 2d$, then there exists $\pi \in \mathcal{P}$ such that $\delta(h, \pi) = \delta_h > 0$. Notice also that if $\delta(h, \pi) = \delta_h$, $E \in \mathfrak{B}$, $E \subset E_\pi^h$ and $\mu(E) > 0$, then $O(E) \geqslant h$; thus, E_π^h is uniquely determined up to a set of measure zero by the equation $\delta(h, \pi) = \delta_h$, so we can denote it by E_h. Now observe that if $h_1 < h_2$, then $\mu(E_{h_2} - E_{h_1}) = 0$.

Let $m(E) = (1/2)(l(E) + u(E))$, $E \in \mathcal{A}$, $\mu(E) > 0$.

The proof of the fourth lemma in [2] implies a lemma that we record for future use.

LEMMA 3. *Let $h > 2\gamma > 0$, let $E(h, \gamma) = E_h - E_{h+\gamma} = \{\bigcup_j F_j;\ F_j \in \mathfrak{B},\ 0 < \mu(F_j),\ h \leqslant O(F_j) < h + \gamma\}$, and let $\hat{f}$ denote the best best L_∞ approximation to f by elements of B. Then $\mu(\{x \in F_i; |\hat{f}(x) - m(F_i)| > \gamma/2\}) = 0$; for almost all points $x \in F_i$, $|\hat{f}(x) - m(F_i)| \leqslant \gamma/2$.*

Now we are ready to show that $\{f_i\}$ converges a.e. To this end, let $0 < \varepsilon < 1$. Let $\gamma > 0$ with $4\gamma < \varepsilon$ and let M be the smallest positive integer for which $\varepsilon + M\gamma \geqslant 1$. Reintroduce the index i to the notation (e.g. for $\mathfrak{B}_i$, E_h becomes $E_{i,h}$). Notice that $E_{i,h} \supset E_{j,h}$, $i \leqslant j$, $h > 0$. For $h > 0$, let $E_h = \bigcap_i E_{i,h}$ and let $D_{i,h} = E_{i,h} - E_h$. Let $\eta > 0$ satisfy the inequality $2\eta(M + 1) < \varepsilon$, and let n be a positive integer such that $\mu(D_{n,\varepsilon+k\gamma}) < \eta$, $k = 0, 1, \ldots, M$. Let $D_n = \bigcup_{k=0}^M D_{n,\varepsilon+k\gamma}$, and let $G_n = E_{n,\varepsilon} - D_n$. Then $\{\Omega - E_{n,\varepsilon}, G_n, D_n\}$ is a partition of Ω. Lemma 1 and Theorem 2 of [2] imply that $|f_i - f| < \varepsilon/2$ on $(\Omega - E_{i,\varepsilon}) \supset (\Omega - E_{n,\varepsilon})$, $i \geqslant n$. Notice that $\mu(D_n) < \varepsilon$. If we show that, for $i \geqslant n$, $|f_i - f_n| < \varepsilon/2$ on G_n, then we will have shown that for each $\varepsilon > 0$ there is a set $D = D_n$ and a positive integer n such that $\mu(D) < \varepsilon$ and $|f_i(x) - f_j(x)| < \varepsilon$, $i, j \geqslant n$, $x \notin D$; since this latter situation is equivalent to a.e. convergence of the sequence $\{f_i\}$ we will be done. So, let $S_{i,k} = (E_{i,\varepsilon+k\gamma} - E_{i,\varepsilon+(k+1)\gamma})$. For $n \leqslant j \leqslant i$, $(S_{j,k} - S_{i,k}) \subset (E_{n,\varepsilon+k\gamma} - E_{\varepsilon+k\gamma})$ because $E_{j,h} \supset E_{i,h}$, $h > 0$. Now observe that

$$E_{n,\varepsilon} = \bigcup_{k=0}^{M} S_{n,k} = G_{n,i} \cup H_{n,i},$$

where $G_{n,i} = \bigcup_{k=0}^M (S_{n,k} - S_{i,k}) \subset D_n$ and $H_{n,i} = \bigcup_{k=0}^M (S_{n,k} \cap S_{i,k}) \supset G_n$. Since $H_{n,i} \supset G_n$, it suffices to show that $|f_n - f_i| < \varepsilon/2$ on $H_{n,i}$, $i > n$, as follows. Fix k to simplify the notation and let $h = \varepsilon + k\gamma$. Then

$$S_{n,k} = \bigcup_l \{F_{n,l};\ \mu(F_{n,l}) > 0, F_{n,l} \in \mathfrak{B}_n, h \leqslant O(F_{n,l}) < h + \gamma\},$$

and

$$S_{i,k} = \bigcup_m \{F_{i,m};\ \mu(F_{i,m}) > 0, F_{i,m} \in \mathfrak{B}_m, h \leqslant O(F_{i,m}) < h + \gamma\}.$$

So $S_{n,k} \cap S_{i,k} = \bigcup_{l,m}(F_{n,l} \cap F_{i,m})$. Fix i, l, m and let H denote $F_{n,l} \cap F_{i,m}$. Observe that $H \in \mathcal{B}_i$. Suppose $\mu(H) > 0$; then $h \leqslant O(H) \leqslant \max\{O(F_{n,l}), O(F_{i,m})\} < h + \gamma$ because $H \subset E_{i,h}$. A computation verifies that $|m(F_{n,l}) - m(F_{i,m})| < \gamma/2$. Finally we apply Lemma 3 and obtain $|f_n(x) - f_i(x)| \leqslant |f_n(x) - m(F_{n,l})| + |m(F_{n,l}) - m(F_{i,m})| + |m(F_{i,m}) - f_i(x)| < 2\gamma < \varepsilon/2$ for $x \in H$. Thus, the following theorem is established.

THEOREM. *Let $f \in A$. Then the sequence $\{f_i\}$ of best best L_∞-approximations to f is bounded by $\|f\|_\infty$ and converges a.e.*

We conclude with an example to illustrate the fact that $\{f_n\}$ need not converge to f_∞.

EXAMPLE. Let $\Omega = [0, 1)$, $\mathcal{A}$ be the Borel sets in Ω, μ denote Lebesgue measure, and let $\mathcal{B}_n$ be generated by $\{[(i-1)/2^n, i/2^n);\ 1 \leqslant i \leqslant 2^n\}$. Let E be a countable union of closed subsets of Ω such that if $0 \leqslant u < v < 1$, then both E and $\Omega - E$ intersect (u, v) in a set of positive measure (cf. [3, p. 59]). Let f be the indicator function I_E of E (i.e., $f(x) = 1$ if $x \in E$ and $f(x) = 0$ if $x \in \Omega - E$). Then $f = f_\infty$; but $f_n \equiv 1/2$, $n = 1, 2, \ldots$.

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