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LIPSCHITZ FUNCTIONS AND SPECTRAL SYNTHESIS

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ABSTRACT. An S-set in the circle group T is a closed subset S of T for which $\overline{j(S)} = k(S)$. We construct a non-S-set S satisfying

$$\bigcup_{\alpha>0} \mathrm{Lip}_\alpha(T) \cap k(S) \subset \overline{j(S)}.$$

Thus $\mathrm{Lip}_\alpha(T) \cap A(T)$ is not a big enough part of $A(T)$ to test the synthesizability of a given closed subset of T.

1. Introduction. Let $A(T)$ denote the Banach algebra of functions with absolutely convergent Fourier series. For a closed subset S of T, we define

$$k(S) = \{f \in A(T): f = 0 \text{ on } S\}$$

and

$$j(S) = \{f \in k(S): f = 0 \text{ on a neighborhood of } S\}.$$

A closed subset S of T is called an S-set if the closure of $j(S)$ in $A(T)$ equals $k(S)$. For notations and basic facts about spectral synthesis, we refer the reader to [1].

It is well known that functions in the Lipschitz space $\mathrm{Lip}_{1/2}(T)$ are synthesizable, in the sense that

$$\mathrm{Lip}_{1/2}(T) \cap k(S) \subset \overline{j(S)},$$

for any closed subset S of T. On the other hand, for each α less than $\frac{1}{4}$, there are nonsynthesizable functions in $\mathrm{Lip}_\alpha(T) \cap A(T)$ (see [2]).

In [6], D. J. Newman shows that, for any closed set S of Lebesgue measure 0, there is a constant $\alpha > 0$ such that

$$\mathrm{Lip}_\alpha(T) \cap k(S) \subset \overline{j(S)}. \tag{1}$$

Using his condition on α, one can see that, for some S, the condition (1) holds for every positive α, that is,

$$\left(\bigcup_{\alpha>0} \mathrm{Lip}_\alpha(T)\right) \cap k(S) \subset \overline{j(S)}. \tag{2}$$

Since $\mathrm{Lip}_\alpha(T) \cap A(T)$, when $0 < \alpha < \frac{1}{4}$, is big enough to contain nonsynthesizable functions, one might guess that the condition (2) is strong enough to imply that S is an S-set.

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In this note we shall construct (Theorem 1) a non-S-set that satisfies condition (2), disproving the above conjecture. Our method can be generalized to get a stronger result (Theorem 2).

2. Lemmas. In this section, let S denote any closed set of Lebesgue measure 0. The complement $S^c = T - S$ is a countable union of open intervals. Write $S^c = \bigcup_{n=0}^{\infty} I_n$, where $\varepsilon_n = |I_n|$, the length of I_n, for $n = 0, 1, 2, \ldots$. Thus $2\pi = \sum_{n=0}^{\infty} \varepsilon_n$.

First of all, we quote a lemma from [6].

Lemma 1. *If* $\sum \varepsilon_n^{\alpha} |\log \varepsilon_n| < \infty$, *then* (1) *holds.*

Note that if $\sum \varepsilon_n^{\alpha} < \infty$ for every α then an easy calculation shows that $\sum \varepsilon_n^{2\alpha} |\log \varepsilon_n| < \infty$. Thus (2) holds for S. One interesting observation is that, under this condition, condition (2) holds for every closed subset of S, as we shall see below.

Lemma 2. *Let S be as above and S' be a closed subset. Write* $S^c = \bigcup_{n=0}^{\infty} I_n$, $(S')^c = \bigcup_{n=0}^{\infty} J_n$, *where* $|I_n| = \varepsilon_n$ *and* $|J_n| = \varepsilon_n'$. *If* $\sum \varepsilon_n^{\alpha} < \infty$ *and* $0 < \alpha < 1$, *then* $\sum (\varepsilon_n')^{\alpha} < \infty$.

Proof. Notice that, for every sequence $\{x_n\}$ of positive real numbers, $\sum_{n=0}^{\infty} x_n^{\alpha} \geqslant (\sum x_n)^{\alpha}$.

Since $S' \subset S$, $S^c \subset (S')^c$ so that each I_n is contained in exactly one of the J_k's. Now it follows that

$$\varepsilon_k' = |J_k| = \left| (J_k \cap S) \cup \left[\bigcup_{n=0}^{\infty} (J_k \cap I_n) \right] \right| = \sum_{n=0}^{\infty} |J_k \cap I_n| = \sum_{I_n \subset J_k} \varepsilon_n.$$

Thus

$$\sum_{n=0}^{\infty} \varepsilon_n^{\alpha} = \sum_{k=0}^{\infty} \left(\sum_{I_n \subset J_k} \varepsilon_n^{\alpha} \right) \geqslant \sum_{k=0}^{\infty} \left(\sum_{I_n \subset J_k} \varepsilon_n \right)^{\alpha} = \sum_{k=0}^{\infty} (\varepsilon_k')^{\alpha}.$$

We use these lemmas to prove the following lemma.

Lemma 3. *Let S be as above. If* $\sum \varepsilon_n^{\alpha} < \infty$ *for every* $\alpha > 0$, *then every closed subset S' of S satisfies* (2).

3. Construction. The first part of our construction will be that of Kahane and Salem [4, p. 13], slightly modified to get a set of Lebesgue measure 0.

Take any sequence $\{t_k\}_{k=0}^{\infty}$ of positive real numbers so that $\sum_0^{\infty} 2^k t_k \leqslant 2\pi$ and $\sum_0^{\infty} 2^k t_k^{\alpha} < \infty$ for each $\alpha > 0$. For example, one may take $t_k = 2^{-k^2}$, $k = 0, 1, 2, \ldots$. Then

$$\sum_0^{\infty} 2^k 2^{-k^2} = 1 + \sum_1^{\infty} 2^{k-k^2} \leqslant 1 + \sum_0^{\infty} 2^{-k} \leqslant 2\pi$$

and $\sum_0^{\infty} 2^k (2^{-k^2})^{\alpha} = \sum_0^{\infty} 2^{-\alpha k^2 + k} < \infty$ by the ratio test.

Put $\varepsilon_n = t_k$ if $2^k \leqslant n < 2^{k+1}$, $k = 0, 1, 2, \ldots$, and $\varepsilon_0 = 2\pi - \sum_1^{\infty} \varepsilon_n$.

Let $I_0 = (\sum_1^\infty \varepsilon_n, 2\pi)$, and

$$E_0 = E_0^1 = \left[0, \sum_1^\infty \varepsilon_n\right] = T - I_0.$$

Note that $|I_0| = 2\pi - \sum_1^\infty \varepsilon_n = \varepsilon_0$. Let I_1 be the open interval of length ε_1 and having center at $\frac{1}{2}\sum_1^\infty \varepsilon_n$, i.e., at the center of the interval E_0^1. Then $E_0^1 - I$ is the union of two closed intervals. Let E_1^1 denote the left interval and E_1^2 the right one. Put $E_1 = E_1^1 \cup E_1^2$. Notice that

$$|E_1^1| = |E_1^2| = \frac{1}{2}\sum_2^\infty \varepsilon_n \quad \text{and} \quad E_1 \subset E_0.$$

Suppose we have defined E_k as a disjoint union of closed intervals $E_k^1, \ldots, E_k^{2^k} \subset E_{k-1}$ so that $|E_k^1| = |E_k^2| = \cdots = |E_k^{2^k}|$. We define E_{k+1} as follows.

Given $j = 1, 2, \ldots, 2^k$, let I_{2^k+j-1} be the open interval of length $\varepsilon_{2^k+j-1} = t_k$ and having the same center as E_k^j. Then $E_k^j - I_{2^k+j-1}$ is the union of two closed intervals of the same length, say E_{k+1}^{2j-1} and E_{k+1}^{2j}, the left interval and the right one, respectively. Put $E_{k+1} = \bigcup_{j=1}^{2^{k+1}} E_{k+1}^j$. Then $S = \bigcap_{n=0}^\infty E_k$ is a perfect set of measure 0. Also, by our choice,

$$S^c = \bigcup_0^\infty I_n, \qquad |I_n| = \varepsilon_n$$

and $\sum \varepsilon_n^\alpha = \sum 2^k t_k^\alpha < \infty$, for each positive α.

Next we shall show that S is a "perfect symmetric" set of the type employed in Kahane and Katznelson's paper [3], that is,

$$S = \left\{ r_0 + \sum_1^\infty \delta_k r_k : \delta_k = \pm 1 \right\},$$

for some suitable sequence $\{r_k\}$ of real numbers. This fact seems to be well known, but the author could not find its proof.

Put $r_0 = \frac{1}{2}|E_0^1|$, and

$$r_k = \tfrac{1}{2}\left(|E_{k-1}^1| - |E_k^1|\right),$$

for $k \geq 1$. Then r_0 is the midpoint of E_0^1, and $r_0 + \sum_1^n \delta_k r_k$ is that of some E_n^j, for $n \geq 1$.

Indeed, suppose $r_0 + \sum_1^n \delta_k r_k$ is the midpoint of E_n^j. If $\delta_{n+1} = 1$, the midpoint of E_{n+1}^{2j} is $(r_0 + \sum_1^n \delta_k r_k) + (\frac{1}{2}|I_{2^{n+1}+j-1}| + \frac{1}{2}|E_{j+1}^{2j}|)$. (Remember that the first term is the midpoint of E_n^j and that the second is the distance between midpoints of E_n^j and E_{n+1}^{2j}.) But $|E_n^j| = |I_{2^{n+1}+j-1}| + 2|E_{n+1}^{2j}|$, so that

$$\tfrac{1}{2}|I_{2^{n+1}+j-1}| + \tfrac{1}{2}|E_{n+1}^{2j}| = \tfrac{1}{2}\left(|E_n^j| - |E_{n+1}^{2j}|\right) = r_k.$$

Hence the midpoint of E_{n+1}^{2j} is $r_0 + \sum_1^{n+1} \delta_k r_k$. Similarly, if $\delta_{n+1} = -1$, then $r_0 + \sum_1^{n+1} \delta_k r_k$ is the midpoint of E_{n+1}^{2j-1}.

Given any $r_0 + \sum_1^\infty \delta_k r_k$ and $n \geq 1$, $r_0 + \sum_1^n \delta_k r_k$ is the midpoint of some E_n^j, and

$$\left| \sum_{k=n+1}^\infty \delta_k r_k \right| \leq \sum_{n+1}^\infty r_k = \frac{1}{2}\sum_{n+1}^\infty \left(|E_{k-1}^1| - |E_k^1|\right) = \tfrac{1}{2}|E_n^1| = \tfrac{1}{2}|E_n^j|.$$

Thus $r_0 + \Sigma_1^\infty \delta_k r_k = (r_0 + \Sigma_1^n \delta_k r_k) + \Sigma_{n+1}^\infty \delta_k r_k \in E_n^j \subset E_n$ and hence

$$r_0 + \sum_1^\infty \delta_k r_k \in S.$$

Conversely, let $x \in S$ be arbitrary. For $n \geqslant 1$, since $x \in E_n$, $x \in E_n^{j_n}$ for some j_n. But $E_n^{j_n}$ contains only $E_{n+1}^{2j_n-1}$ and $E_{n+1}^{2j_n}$. Thus j_{n+1} is either $2j_n - 1$ or $2j_n$. Define

$$\delta_n = \begin{cases} -1, & \text{if } j_{n+1} = 2j_n - 1, \\ 1, & \text{if } j_{n+1} = 2j_n. \end{cases}$$

Now, by using induction once more, one can see that $r_0 + \Sigma_1^n \delta_k r_k$ is the midpoint of $E_n^{j_n}$. So it follows that

$$\left| x - \left(r_0 + \sum_1^n \delta_k r_k \right) \right| < \tfrac{1}{2}|E_n^{j_n}| = \tfrac{1}{2}|E_n^1| \to 0,$$

as $n \to \infty$. Thus it follows that $x = r_0 + \Sigma_1^\infty \delta_k r_k$, which proves our claim.

Since no set of the form $\{r_0 + \Sigma_1^\infty \delta_k r_k : \delta_k = \pm 1\}$ is a set of spectral resolution [3], S contains a closed subset S' which is not an S-set. But, by Lemma 3, the condition (2) holds for S'. If we summarize these results, we get

THEOREM 1. *There is a non-S-set S, satisfying*

$$\left(\bigcup_{\alpha > 0} \operatorname{Lip}_\alpha(T) \right) \cap k(S) \subset \overline{j(S)}.$$

4. Remarks. Our result shows that the space $\operatorname{Lip}_\alpha(T) \cap A(T)$ is not big enough to test the synthesizability of a given closed set.

Let φ be any nondecreasing continuous function on $[0, 2\pi]$ satisfying the following conditions:

$\varphi(0) = 0$, $\varphi(x) > 0$ if $x > 0$, $\varphi(x)/x$ is decreasing on $(0, 2\pi]$ and $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$ for positive x and y.

Let $V_\varphi = \{f \in C(T): \|f_\tau - f\|_{C(T)} \leqslant C\varphi(|\tau|)$, for some $C > 0\}$, where $f_\tau(t) = f(t - \tau)$ is the translation of f by τ. Notice that $\operatorname{Lip}_\alpha(T) = V_\varphi$ for $\varphi(t) = t^\alpha$.

If we use Theorem 3 of [6] instead of Lemma 1, we can modify our proofs to get the following theorem, which says, roughly, that the space of functions satisfying some regularity condition is not big enough to test the synthesizability of a set.

THEOREM 2. *Let φ be as above. Then there is a non-S-set S, satisfying*

$$V_\varphi \cap k(S) \subseteq \overline{j(S)}.$$

On p. 88 of [5], T. W. Körner constructed, for every continuous increasing function H on $[0, \infty)$ with $H(0) = 0$, a non-S-set S having, along with other properties, Hausdorff H-measure 0. If we take the function $H(t) = -1/\log t$, $0 \leqslant t < \frac{1}{3}$, we get $t^\alpha = o(H(t))$ as $t \to 0$, $\alpha > 0$, and hence the Hausdorff α-measure $H^\alpha(S) = 0$ (see [4, p. 26]). Thus, the Hausdorff dimension, $\alpha(S)$, of S is 0, so that S satisfies the condition (2) (see [6]). This gives an alternate proof of our Theorem 1.

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