

## ON STRONG UNICITY OF $L_1$ -APPROXIMATION

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**ABSTRACT.** Let  $C_1$  be the space of continuous functions on  $[0, 1]$  with norm  $\|f\| = \int_0^1 |f(x)| dx$ , and let  $G \subset C_1$  be a finite dimensional unicity subspace, i.e. any  $f \in C_1$  possesses a unique best approximation out of  $G$ . Consider an arbitrary  $f \in C_1$  such that  $0$  is its best approximation. Then for any  $0 < \delta < \delta_0$  and  $\tilde{g} \in G$  with  $\|f - \tilde{g}\| < \|f\| + \delta$  it follows that  $\|\tilde{g}\| < K w_f^*(\delta)$ , where  $w_f^*(h) = (w_f^{-1}(h)h)^{-1}$  and  $w_f(h)$  denotes the modulus of continuity of  $f$ . ( $-1$  is used to denote the inverse function.)

**Introduction.** Let  $X$  be a normed linear space and let  $G$  be a finite dimensional unicity subspace of  $X$ ; that is any  $f \in X$  possesses a unique element of best approximation  $g^* \in G$  ( $\|f - g^*\| = \inf\{\|f - g\| : g \in G\}$ ). If  $0$  is the best approximation of  $f \in X$ , a question of practical interest is that of how fast the “nearly best approximations”  $g \in G$  satisfying  $\|f - g\| < \|f\| + \delta$  approach zero when  $\delta \rightarrow 0$ . Its practical interest is connected with the study of rate of convergence of different computational algorithms for best approximation and investigation of continuity of the operator of best approximation.

The study of strong unicity was initiated by D. Newman and H. Shapiro [8] for Chebyshev approximation. Their results inspired a wide investigation of this question for different functional spaces. (See also a recent paper of R. Wegmann [12].)

The purpose of the present note is to solve this problem for  $L_1$ -approximation. By a well-known result of M. Krein (see [9, p. 230]) there are no finite dimensional unicity subspaces in  $L_1$ . Therefore we consider the space  $X = C_1$  of continuous real-valued functions on  $I = [0, 1]$  with norm  $\|f\| = \int_I |f(x)|$  (Lebesgue integral). The classical Jackson-Krein theorem states that any Haar subspace of  $C_1$  is a unicity subspace. The strong unicity of  $L_1$ -approximation in the case when  $G$  is a Haar subspace of  $C_1$  was studied by B. O. Björnestrål [1], [2]; he gave the exact rate of strong unicity under this condition. Recently, P. V. Galkin [5] and H. Strauss [10] discovered that Haar subspaces are not the only unicity subspaces of  $C_1$ . They proved that splines with fixed knots also satisfy this property. In [3] an analogous result was given for a special class of splines. These results raise the question of estimating the rate of strong unicity of  $L_1$ -approximation for arbitrary unicity subspaces. It turned out that estimations given by B. O. Björnestrål for Haar subspaces can be extended to arbitrary unicity subspaces. This extension needs deeper considerations involving certain ideas of E. W. Cheney and D. E. Wulbert [4] concerning characterization of unicity subspaces of  $C_1$ .

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Let us denote by  $w_f(h) = \max\{|f(x_1) - f(x_2)|: x_1, x_2 \in I, |x_1 - x_2| \leq h\}$  the modulus of continuity of  $f \in C_1$ . Then set  $w_f^*(h) = (w_f^{-1}(h)h)^{-1}$ , where  $F^{-1}(x) = \min\{y: F(y) = x\}$  denotes the inverse function.

**THEOREM.** *Let  $G$  be a finite dimensional unicity subspace of  $C_1$ . Consider an arbitrary  $f \in C_1$  such that 0 is its best approximation. Then for any  $0 < \delta \leq w_f(1)$  and  $\tilde{g} \in G$  satisfying  $\|f - \tilde{g}\| \leq \|f\| + \delta$  it follows that*

$$(1) \quad \|\tilde{g}\| \leq c_1 w_f^*(\delta),$$

where the constant  $c_1$  depends only on  $f$  and  $G$ . Moreover this estimation cannot be improved in general.

**PROOF OF THEOREM.** We shall need some well-known characterizations of best  $L_1$ -approximation.

The following statements are equivalent (see [9, p. 46]):

- (i) 0 is the best approximation of  $f$ ;
- (ii)  $|\int_{I \setminus Z(f)} g \operatorname{sign} f| \leq \int_{Z(f)} |g|$  ( $g \in G$ );
- (iii) there exists a measurable function  $\varphi_f$  on  $Z(f)$  such that  $|\varphi_f| \leq 1$  on  $Z(f)$  and

$$\int_{I \setminus Z(f)} g \operatorname{sign} f + \int_{Z(f)} g \varphi_f = 0 \quad (g \in G).$$

Here and throughout the paper  $Z(f) = \{x \in I: f(x) = 0\}$ . For a given  $f$  we always fix an arbitrary  $\varphi_f$  satisfying (iii) and set

$$f^*(x) = \begin{cases} \operatorname{sign} f(x), & x \in I \setminus Z(f), \\ \varphi_f(x), & x \in Z(f). \end{cases}$$

Furthermore, set  $O_h(x) = I \cap (x - h, x + h)$ ;

$Z_1(f) = \{x \in I: \mu(O_h(x) \cap \{x \in I: |f^*(x)| < 1\}) > 0 \text{ for any } h > 0\}$ ;

$Z_2(f) = \{x \in I: \mu(O_h(x) \cap \{x \in I: f^*(x) = \gamma\}) > 0$

for any  $h > 0$  and  $\gamma = 1$  or  $-1\}$

( $\mu(A)$  denotes the Lebesgue measure of  $A$ );  $Z^*(f) = Z_1(f) \cup Z_2(f)$ . It can be easily verified that  $Z_1(f)$  and  $Z_2(f)$  are closed subsets of  $I$ ; hence  $Z_1(f)$ ,  $Z_2(f)$  and  $Z^*(f)$  are compact. This implies that for any given  $h > 0$

$$(2) \quad \inf_{x \in Z_1(f)} \int_{O_h(x)} (1 - |f^*|) = \alpha_f(h) > 0$$

and

$$(3) \quad \min_{\gamma = \pm 1} \inf_{x \in Z_2(f)} \mu(O_h(x) \cap \{x \in I: f^*(x) = \gamma\}) = \beta_f(h) > 0.$$

Inequalities (2) and (3) are obvious consequences of the fact that continuous functions on compact sets achieve their minimal value.

We shall need the following lemma which is a modification of a result of Cheney and Wulbert (see [4, Theorem 23]).

LEMMA. Let  $G$  be a finite dimensional unicity subspace of  $C_1$ . If 0 is the best approximation of  $f \in C_1$ , then

$$(4) \quad \sup_{x \in Z^*(f)} |g(x)| \geq c_2 \sup_{x \in I} |g(x)| \quad (g \in G),$$

where the constant  $c_2$  is independent of  $g$ .

PROOF OF THE LEMMA. By the equivalence of norms in finite dimensional spaces we obtain that it is enough to prove that 0 is the only element of  $G$  vanishing on  $Z^*(f)$ . Assume the contrary. Then there exists  $g_1 \in G \setminus \{0\}$  such that  $Z^*(f) \subset Z(g_1)$ . Let  $(a, b)$  be an arbitrary interval on which  $g_1$  does not vanish. Then for any  $x \in (a, b)$ ,  $x \notin Z^*(f) = Z_1(f) \cup Z_2(f)$ . Hence there exists an  $h_x > 0$  such that  $f^* = \gamma_x$  a.e. on  $O_{h_x}(x)$ , where  $\gamma_x = 1$  or  $-1$ . By standard compactness arguments we obtain that  $f^* = \gamma$  a.e. on  $(a, b)$ , where  $\gamma = 1$  or  $-1$ . Then we can easily construct a continuous function  $p \in C_1$  such that  $|p| = |g_1|$  on  $I$  and  $\text{sign } p = f^*$  a.e. on  $I \setminus Z(g_1) = I \setminus Z(p)$ . This and the definition of  $f^*$  yield

$$\left| \int_{I \setminus Z(p)} g \text{sign } p \right| = \left| \int_{I \setminus Z(p)} g f^* \right| = \left| \int_{Z(p)} g f^* \right| < \int_{Z(p)} |g| \quad (g \in G).$$

Thus 0 is the best approximation of  $p$ . But for any  $0 < \theta < 1$ ,  $Z(p - \theta g_1) = Z(p)$  and  $\text{sign}(p - \theta g_1) = \text{sign } p$  on  $I \setminus Z(p)$ . This implies that  $\theta g_1$  is also a best approximation of  $p$ ; hence we have arrived at a contradiction. The lemma is proved.

Now we are able to prove the theorem. Take an arbitrary  $\tilde{g} \in G \setminus \{0\}$  such that  $\|f - \tilde{g}\| \leq \|f\| + \delta$  ( $0 < \delta \leq w_f(1)$ ). Since 0 is the best approximation of  $f \in C_1$  we have

$$\begin{aligned} \delta &\geq \|f - \tilde{g}\| - \|f\| \\ (5) \quad &= \int_{I \setminus Z(f)} (f - \tilde{g}) \{ \text{sign}(f - \tilde{g}) - \text{sign } f \} + \int_{Z(f)} \{ |\tilde{g}| + \tilde{g} \varphi_f \} \\ &= 2 \int_{A(f, \tilde{g})} |f - \tilde{g}| + \int_{Z(f)} \{ |\tilde{g}| + \tilde{g} \varphi_f \}, \end{aligned}$$

where  $A(f, \tilde{g}) = \{x \in I: 0 < f(x) < \tilde{g}(x) \text{ or } \tilde{g}(x) < f(x) < 0\}$ . By (4) and the compactness of  $Z^*(f)$ , there exists an  $\tilde{x} \in Z^*(f)$  such that

$$(6) \quad \tilde{\gamma} \tilde{g}(\tilde{x}) \geq c_2 \sup_{x \in I} |\tilde{g}(x)| = c_2 \|\tilde{g}\|_C \quad (\tilde{\gamma} = 1 \text{ or } -1).$$

(Here and in what follows  $\|\cdot\|_C$  denotes the supremum norm on  $I$ ,  $c_i$  ( $i = 1, 2, 3, \dots$ ) will denote positive constants depending just on  $f$  and  $G$ .) Furthermore, it can be easily proved that, for any  $g \in G$ ,  $w_g(h) \leq \|g\|_C \bar{w}(h)$ , where  $\bar{w}(h)$  is a modulus of continuity depending only on  $G$ . This and (6) imply that

$$(7) \quad \tilde{\gamma} \tilde{g}(x) \geq (c_2/2) \|\tilde{g}\|_C$$

for any  $x \in O_{c_3}(\tilde{x})$ , where  $c_3 = \bar{w}^{-1}(c_2/2)$ . Since  $\tilde{x} \in Z^*(f)$ ,  $\tilde{x}$  belongs to either  $Z_1(f)$  or  $Z_2(f)$ . We shall consider these two cases separately.

Case 1.  $\tilde{x} \in Z_1(f)$ . Set  $F = O_{c_3}(\tilde{x}) \cap \{x \in I: |f^*(x)| < 1\}$ . Since  $|f^*| = 1$  on  $I \setminus Z(f)$ ,  $F \subset Z(f)$ . Thus we have by (5), (7) and (2)

$$\begin{aligned} \delta &\geq \int_{Z(f)} \{|\tilde{g}| + \tilde{g}\varphi_f\} > \int_F |\tilde{g}|(1 - |f^*|) = \int_{O_{c_3}(\tilde{x})} |\tilde{g}|(1 - |f^*|) \\ &\geq (c_2/2) \|\tilde{g}\|_C \alpha_f(c_3) > c_4 \|\tilde{g}\|. \end{aligned}$$

Since  $\delta \leq c_5 w_f^*(\delta)$  if  $0 < \delta \leq w_f(1)$  we obtain estimation (1).

Case 2.  $\tilde{x} \in Z_2(f)$ . Then by (3)

$$(8) \quad \mu(O_{c_3}(\tilde{x}) \cap \{x \in I: f^*(x) = \tilde{\gamma}\}) > \beta_f(c_3) = c_6.$$

Set  $B = O_{c_3}(\tilde{x}) \cap \{x \in I \setminus Z(f): \text{sign } f = \tilde{\gamma}\}$ ,  $Q = O_{c_3}(\tilde{x}) \cap \{x \in Z(f): \varphi_f = \tilde{\gamma}\}$ . Then by (8)

$$(9) \quad \mu(B) + \mu(Q) > c_6.$$

Assume at first that  $\mu(Q) > c_6/2$ . Since  $Q \subset Z(f)$ , we have by (5) and (7)

$$\begin{aligned} c_5 w_f^*(\delta) &\geq \delta > \int_{Z(f)} \{|\tilde{g}| + \tilde{g}\varphi_f\} > \int_Q \{|\tilde{g}| + \tilde{g}\varphi_f\} \\ &= 2 \int_Q |\tilde{g}| > c_2 \|\tilde{g}\|_C \mu(Q) > c_7 \|\tilde{g}\|, \end{aligned}$$

which was to be proved. Thus we may assume that  $\mu(Q) < c_6/2$ . Then by (9)

$$(10) \quad \mu(B) > c_6/2.$$

Case 2a.  $\tilde{\gamma}f < (c_2/4) \|\tilde{g}\|_C$  on  $O_{c_3}(\tilde{x})$ . Then (7) implies that  $B \subset A(f, \tilde{g})$ . Thus by (5), (7) and (10) we have

$$\begin{aligned} c_5 w_f^*(\delta) &\geq \delta > 2 \int_{A(f, \tilde{g})} |f - \tilde{g}| > 2 \int_B |f - \tilde{g}| \\ &\geq 2 \int_B \left( \frac{c_2}{2} \|\tilde{g}\|_C - \frac{c_2}{4} \|\tilde{g}\|_C \right) = \frac{c_2}{2} \|\tilde{g}\|_C \mu(B) > c_8 \|\tilde{g}\| \end{aligned}$$

which is again the needed estimation.

Case 2b.  $\tilde{\gamma}f(x^*) \geq (c_2/4) \|\tilde{g}\|_C$  for some  $x^* \in O_{c_3}(\tilde{x})$ . Since  $\tilde{x} \in Z_2(f) \subset Z(f)$ ,  $f(\tilde{x}) = 0$ . Let us assume that  $x^* > \tilde{x}$ . (The opposite case can be considered analogously.) Then we can choose points  $\tilde{x} < x_1 < x_2 < x^*$  such that  $f(x_1) = 0$ ,  $\tilde{\gamma}f(x_2) = (c_2/4) \|\tilde{g}\|_C$  and  $0 < \tilde{\gamma}f < (c_2/4) \|\tilde{g}\|_C$  on  $(x_1, x_2)$ . Evidently

$$(11) \quad x_2 - x_1 \geq w_f^{-1} \left( \frac{c_2}{4} \|\tilde{g}\|_C \right).$$

Moreover, since  $(x_1, x_2) \subset (\tilde{x}, x^*) \subset O_{c_3}(\tilde{x})$ , (7) implies that  $(x_1, x_2) \subset A(f, \tilde{g})$ . Hence using (5), (7) and (11) we obtain

$$\begin{aligned} \delta &\geq 2 \int_{x_1}^{x_2} |f - \tilde{g}| \geq 2 \int_{x_1}^{x_2} \left( \frac{c_2}{2} \|\tilde{g}\|_C - \frac{c_2}{4} \|\tilde{g}\|_C \right) \geq \frac{c_2}{2} \|\tilde{g}\|_C (x_2 - x_1) \\ &\geq \frac{c_2}{2} \|\tilde{g}\|_C w_f^{-1} \left( \frac{c_2}{4} \|\tilde{g}\|_C \right) \geq \frac{c_2}{4} \|\tilde{g}\|_C w_f^{-1} \left( \frac{c_2}{4} \|\tilde{g}\|_C \right), \end{aligned}$$

i.e.  $(c_2/4) \|\tilde{g}\| \leq w_f^*(\delta)$ . The proof of (1) is completed.

The sharpness of estimation (1) can be verified similarly as in [6], since the Haar property was not employed there (see [6, p. 341]).

**Conclusion.** The usual approach to the solution of the  $L_1$ -approximation problem consists in its discretization. In [7] and [11] the rate of convergence of discrete  $L_1$ -approximants was studied for Haar subspaces. These investigations were based on strong unicity type results. The theorem proved above allows us to extend these investigations to any unicity subspaces, thus in particular to splines.

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