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SPECTRAL INCLUSION FOR DOUBLY COMMUTING SUBNORMAL n-TUPLES

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ABSTRACT. Let $S = (S_1, \ldots, S_n)$ be a doubly commuting n-tuple of subnormal operators on a Hilbert space $\mathcal{H}$ and $N = (N_1, \ldots, N_n)$ be its minimal normal extension acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. We show that $\mathrm{Sp}(S, \mathcal{H}) \supset \mathrm{Sp}(N, \mathcal{K})$ and $\mathrm{Sp}(S, \mathcal{H}) \subset \text{p.c.h.}(\mathrm{Sp}(N, \mathcal{K}))$, where Sp denotes Taylor spectrum and p.c.h. polynomially convex hull.

1. Introduction. Let S be a subnormal operator on a Hilbert space $\mathcal{H}$ and N be its minimal normal extension to a Hilbert space $\mathcal{K} \supset \mathcal{H}$. A well-known result of Halmos [5] asserts that $\sigma(S) \supset \sigma(N)$, where σ denotes spectrum. Bram then proved in [2] that $\sigma(S) \subset \text{p.c.h.}(\sigma(N))$, the polynomially convex hull of $\sigma(N)$. (He actually proved more: if U is any bounded component of $\mathbf{C} \setminus \sigma(N)$, then $U \cap \sigma(S) = \varnothing$ or $U \subset \sigma(S)$.)

The question arises as to whether the spectral inclusion holds for commuting n-tuples $S = (S_1, \ldots, S_n)$ of subnormal operators on $\mathcal{H}$. A first comment is in order: not every such n-tuple has a commuting normal extension, i.e., it is not always possible to find a commuting n-tuple $N = (N_1, \ldots, N_n)$ of normal operators on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i|_{\mathcal{H}} = S_i$ for all $i = 1, \ldots, n$ (see [8] for an example). There are a number of conditions that guarantee the existence of such an extension (see for instance [1, 6, 7 and 9]). We shall call the n-tuple S subnormal in case it admits a commuting normal extension. It follows from Bram's paper [2] (combining the corollary on p. 88 with Theorem 8) that any doubly commuting n-tuple $S = (S_1, \ldots, S_n)$ (i.e., $S_iS_j = S_jS_i$ for all i, j and $S_iS_j^* = S_j^*S_i$ for $i \neq j$) of subnormal operators is subnormal. (Ito [6] has extended this further.) Also, it is clear that a subnormal n-tuple has a unique, up to isometric isomorphism, minimal normal extension. The n-tuples to be considered are, therefore, the subnormal ones. We must now agree on the right notion of joint spectrum. First, we need some notation. For an n-tuple $T = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$, let $\sigma_r(T)$ denote the right spectrum of T, that is, $\sigma_r(T) = \{\lambda \in \mathbf{C}^n$: $\sum_{i=1}^n (T_i - \lambda_i)(T_i - \lambda_i)^*$ is not invertible$\}$. If T is commuting and $\mathcal{A}$ is a Banach algebra containing the T_i's in its center, let $\sigma_{\mathcal{A}}(T)$ and $\mathrm{Sp}(T, \mathcal{H})$ denote the spectra of T with respect to $\mathcal{A}$ and $\mathcal{H}$, respectively, i.e., $\sigma_{\mathcal{A}}(T) = \{\lambda \in \mathbf{C}^n$: $\sum_{i=1}^n (T_i - \lambda_i)A_i = I$ cannot be solved for $A_i \in \mathcal{A}\}$. (For a definition of Sp see [4 or 12].)

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Janas has shown in [7] that if S is subnormal with minimal normal extension N and $\mathcal{Q}$ is a maximal abelian Banach algebra containing the S_i's, then $\sigma_{\mathcal{Q}}(S) \supset \sigma(N)$. (As it turns out, there is universal agreement on the right notion of spectrum for a normal n-tuple, since $\sigma_r(N) = \mathrm{Sp}(N, \mathcal{H}) = \sigma_{C^*(N)}(N) = \sigma_B(N)$ for any abelian C^*-algebra B containing the N_i's.) It is a result of Taylor [12, Lemma 2.1] that $\mathrm{Sp}(T, \mathcal{H}) \subset \sigma_{\mathcal{Q}}(T)$ for any Banach algebra $\mathcal{Q}$ whose center contains the T_i's, so that $\mathrm{Sp}(S, \mathcal{H}) \supset \mathrm{Sp}(N, \mathcal{H})$ is perhaps the appropriate inclusion to study. One could look for joint spectra smaller than Sp, like those considered by Słodkowski [11]. There are easy examples that show that $\sigma_{\pi,k}$ ($k < n$) will not do; on the other hand, $\sigma_{\delta,0} = \mathrm{Sp}$ for a doubly commuting subnormal n-tuple [4, Corollary 3.8]. (The notation in the last sentence is from [11].) We have posed in [4] the following question: Does $\mathrm{Sp}(S, \mathcal{H}) \supset \mathrm{Sp}(N, \mathcal{H})$? In this paper we give an affirmative answer where S is doubly commuting. Using a result of Janas we also prove that $\mathrm{Sp}(S, \mathcal{H}) \subset \text{p.c.h.}(\mathrm{Sp}(N, \mathcal{H}))$. Our proof is based on a theorem of Bram's on the commutant of the C^*-algebra generated by a subnormal operator, a basic estimate for the left spectrum of an n-tuple, the functional calculus for normal n-tuples and our characterization of Sp for doubly commuting n-tuples of hyponormal operators (as σ_r).

2. A basic fact about the left spectrum. Let $T = (T_1, \ldots, T_n)$ be an n-tuple (not necessarily commuting) of operators $\mathcal{H}$ and $\sigma_l(T)$ be the left spectrum of T, that is,

$$\sigma_l(T) = \left\{\lambda \in \mathbf{C}^n \colon \sum_{i=1}^{n} (T_i - \lambda_i)^*(T_i - \lambda_i) \text{ is not invertible}\right\}.$$

Let

$$\delta(T) = \inf\left\{ \|Tx\| = \left(\sum_{i=1}^{n} \|T_i x\|^2\right)^{1/2} : \|x\| = 1\right\}$$

and

$$m_l(T) = \inf\left\{ |\lambda| = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2} : \lambda \in \sigma_l(T)\right\}.$$

The following lemma is probably well known among the specialists. We include a proof for the sake of completeness (see [10] for a different proof).

LEMMA 1. *For an arbitrary n-tuple T, $m_l(T) \geqslant \delta(T)$.*

PROOF. Let $\lambda \in \mathbf{C}^n$ and $x \in \mathcal{H}$, $\|x\| = 1$. Then

$$\sum_{i=1}^{n} \|(T_i - \lambda_i)x\|^2 = \sum_{i=1}^{n} \|T_i x\|^2 + \sum_{i=1}^{n} |\lambda_i|^2 - 2\sum_{i=1}^{n} \mathrm{Re}(T_i x, \lambda_i x)$$

and

$$\left|\sum_{i=1}^{n} \mathrm{Re}(T_i x, \lambda_i x)\right| \leqslant \sum_{i=1}^{n} |\lambda_i| \, \|T_i x\| \leqslant \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|T_i x\|^2\right)^{1/2}.$$

Thus,

$$\sum_{i=1}^{n} \|(T_i - \lambda_i)x\|^2 \geq \left[\left(\sum_{i=1}^{n} \|T_i x\|^2 \right)^{1/2} - \left(\sum_{i=1}^{n} |\lambda_i|^2 \right)^{1/2} \right]^2.$$

Therefore, if $|\lambda| < \delta(T)$ then $\sum_{i=1}^{n}\|(T_i - \lambda_i)x\|^2 \geq (\delta(T) - |\lambda|)^2$, so that $\lambda \notin \sigma_l(T)$, from which the result follows.

LEMMA 2. *Let N be a commuting n-tuple of normal operators. Then $m_l(N) = \delta(N)$.*

PROOF. $C^*(N_1, \ldots, N_n) \cong C(\sigma_l(N))$.

3. Bram's commutant theorem.

LEMMA 3 (THEOREM 8 IN [2]; SEE ALSO [3, CHAPTER IV]). *Let S be a subnormal operator on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K} \supset \mathcal{H}$. Let $C^*(N)'$, $C^*(S)'$ and $C^*(P)'$ denote the commutants of the C^*-algebras generated by N, S and the projection P of $\mathcal{K}$ onto $\mathcal{H}$ ($P = P_{\mathcal{H}}$). The map*

$$C^*(N)' \cap C^*(P)' \xrightarrow{\Phi} C^*(S)', \qquad T \to T|_{\mathcal{H}},$$

*is an isometric *-isomorphism. Moreover, if $Q \in C^*(S)'$ is a projection, then $\Phi^{-1}(Q)$ is the projection on $\mathcal{K}$ whose range is the closed linear span of the family $\{N^{*n}x\colon x \in Q\mathcal{H}, n \geq 0\}$, so that $N|_{\Phi^{-1}(Q)}$ is the minimal normal extension of $S|_Q$.*

4. The main result. The following lemma is the keystone for our proof of the spectral inclusion.

LEMMA 4. *Let S be a subnormal operator on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K} \supset \mathcal{H}$. Let H be a positive operator in $C^*(S)'$ and $K = \Phi^{-1}(H)$ the (positive) operator given by Bram's theorem. Assume that $0 \notin \sigma_r(S, H)$. Then $0 \notin \sigma_r(N, \mathcal{K})$.*

PROOF. By definition of σ_r, we know that $SS^* + H^2$ is invertible, say $SS^* + H^2 \geq 3\varepsilon$ for some $\varepsilon > 0$. Let the positive numbers t_k and projections $Q_k \in C^*(S)'$ ($k = 1, \ldots, m$) be chosen so that

(i) $\sum_{k=1}^{m} Q_k = I$,
(ii) $Q_k Q_l = 0$ if $k \neq l$, and
(iii) $\|H^2 - \sum_{k=1}^{m} t_k^2 Q_k\| < \varepsilon$.

Then

$$SS^* + \sum_{k=1}^{m} t_k^2 Q_k \geq 2\varepsilon. \tag{$*$}$$

Since the ranges of the Q_k's reduce S (all k), are pairwise orthogonal and span $\mathcal{H}$, we can define $S_k = S|_{Q_k\mathcal{H}}$ acting on $Q_k\mathcal{H}$ and write ($*$) as

$$\bigoplus_{k=1}^{m} (S_k S_k^* + t_k^2) \geq 2\varepsilon.$$

Thus, for each k, $S_k S_k^* + t_k^2 \geq 2\varepsilon$, or $\|S_k^* x\|^2 + t_k^2 \geq 2\varepsilon$, $x \in Q_k\mathcal{H}$, $\|x\| = 1$. In the notation of Lemma 1 this is $\delta(S_k^*, t_k) \geq \sqrt{2\varepsilon}$, so that $m_l(S_k^*, t_k) \geq \sqrt{2\varepsilon}$, too.

Now, by the projection property for the left spectrum,

$$\sigma_l(S_k^*, t_k) = \sigma_l(S_k^*) \times \{t_k\} = \overline{\sigma_r(S_k)} \times \{t_k\}$$

(the horizontal bar denoting complex conjugation). Of course, $\sigma_r(S_k) = \sigma(S_k)$, because S_k is subnormal. Then

$$\sigma_l(S_k^*, t_k) = \overline{\sigma(S_k)} \times \{t_k\} \supset \overline{\sigma(N_k)} \times \{t_k\},$$

by the spectral inclusion theorem for subnormal operators and the fact that $N_k = N|_{\Phi^{-1}(Q_k)\mathcal{K}}$ is the minimal normal extension of S_k. Therefore, $m_l(N_k, t_k) \geqslant m_l(S_k^*, t_k) \geqslant \sqrt{2\varepsilon}$. By Lemma 2, however, $\delta(N_k, t_k) = m_l(N_k, t_k)$, so that $\|N_k x\|^2 + t_k^2 \geqslant 2\varepsilon$, $x \in \Phi^{-1}(Q_k)\mathcal{K}$, $\|x\| = 1$.

Therefore $\bigoplus_{k=1}^m (N_k^* N_k + t_k^2) \geqslant 2\varepsilon$, or

$$(**) \qquad N^*N + \sum_{k=1}^{m} t_k^2 \Phi^{-1}(Q_k) \geqslant 2\varepsilon.$$

From (iii) above and the fact that Φ is an isometry, we get

$$\left\| K^2 - \sum_{k=1}^{m} t_k^2 \Phi^{-1}(Q_k) \right\| < \varepsilon.$$

This last equation combined with $(**)$ gives $N^*N + K^2 \geqslant \varepsilon$, as desired.

5. The spectral inclusion theorem. We need one more lemma before we can prove our theorem.

LEMMA 5 (COROLLARY 3.8 IN [4]). *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on $\mathcal{H}$. Then* $\mathrm{Sp}(T, \mathcal{H}) = \sigma_r(T)$.

THEOREM 1 (SPECTRAL INCLUSION). *Let $S = (S_1, \ldots, S_n)$ be a doubly commuting subnormal n-tuple on $\mathcal{H}$ with minimal normal extension $N = (N_1, \ldots, N_n)$ on $\mathcal{K} \supset \mathcal{H}$. Then* $\mathrm{Sp}(S, \mathcal{H}) \supset \mathrm{Sp}(N, \mathcal{K})$.

PROOF. Assume $n \geqslant 2$. As in the one-variable case, it is enough to show that $0 \notin \mathrm{Sp}(S, \mathcal{H})$ implies $0 \notin \mathrm{Sp}(N, \mathcal{K})$. Now, if $0 \notin \mathrm{Sp}(S, \mathcal{H})$ and $H = (\sum_{i=2}^n S_i S_i^*)^{1/2}$, then (S_1, H) is right invertible. Let $T_1^{(1)} = \text{m.n.e.}(S_1)$ acting on $\mathcal{K}^{(1)} \subset \mathcal{K}$ and Φ_1: $C^*(T_1^{(1)})' \cap C^*(P_{\mathcal{H}})' \to C^*(S_1)'$ be Bram's isomorphism. Let $T_i^{(1)} = \Phi_1^{-1}(S_i)$, $i = 2, \ldots, n$. Notice that $\Phi_1^{-1}(H) = (\sum_{i=2}^n T_i^{(1)} T_i^{(1)*})^{1/2}$ and that each $T_i^{(1)}$ is subnormal; actually, $T_i^{(1)} = N_i|_{\mathcal{K}^{(1)}}$. By Lemma 4, $(T_1^{(1)}, \Phi_1^{-1}(H))$ is right invertible, so that $T^{(1)} = (T_1^{(1)}, \ldots, T_n^{(1)})$ is right invertible, or $0 \notin \mathrm{Sp}(T^{(1)}, \mathcal{K}^{(1)})$, by Lemma 5.

We can now extend $T_2^{(1)}$ to its minimal norm extension $T_2^{(2)}$ on $\mathcal{K}^{(2)} \subset \mathcal{K}$ and repeat the argument so that $0 \notin \mathrm{Sp}(T^{(2)}, \mathcal{K}^{(2)})$. We can continue this process until $T_n^{(n-1)}$ has been extended. Finally, it is clear that $\mathcal{K}^{(n)} = \mathcal{K}$ and $T^{(n)} = N$, so that $0 \notin \mathrm{Sp}(N, \mathcal{K})$, as desired.

REMARK. With the notation as in the preceding proof, notice that we actually proved that

$$\mathrm{Sp}(N, \mathcal{K}) \subset \mathrm{Sp}(T^{(n-1)}, \mathcal{K}^{(n-1)}) \subset \cdots \subset \mathrm{Sp}(T^{(1)}, \mathcal{K}^{(1)}) \subset \mathrm{Sp}(S, \mathcal{H}).$$

THEOREM 2. *Let $S = (S_1, \ldots, S_n)$ be a subnormal n-tuple on $\mathcal{H}$ (not necessarily doubly commuting) and $N = (N_1, \ldots, N_n)$ be its minimal normal extension acting on $\mathcal{K} \supset \mathcal{H}$. Then* $\mathrm{Sp}(S, \mathcal{H}) \subset p.c.h.(\mathrm{Sp}(N, \mathcal{K}))$.

PROOF (SEE JANAS [7, COROLLARY TO THEOREM 5]). Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \operatorname{Sp}(S, \mathcal{H})$ and $P(z_1, \ldots, z_n)$ be a polynomial. Then $P(\lambda) \in P(\operatorname{Sp}(S, \mathcal{H})) = \operatorname{Sp}(P(S), \mathcal{H})$, by the Spectral Mapping Theorem for Taylor spectrum [**13**, Theorem 4.8], so that

$$\begin{aligned} |P(\lambda)| &\leq \sup\{|z| : z \in \sigma(P(S))\} = \|P(S)\| \leq \|P(N)\| \\ &= \sup\{|z| : z \in \sigma(P(N))\} = \sup\{|P(z)| : z \in \operatorname{Sp}(N, \mathcal{K})\}. \end{aligned}$$

Thus $\lambda \in$ p.c.h.$(\operatorname{Sp}(N, \mathcal{K}))$.

COROLLARY. *Let S be a doubly commuting subnormal n-tuple on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K} \supset \mathcal{H}$. Assume that* $\operatorname{Sp}(S, \mathcal{H})$ *is polynomially convex. Then* $\operatorname{Sp}(S, \mathcal{H}) = p.c.h.(\operatorname{Sp}(N, \mathcal{K}))$.

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