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ON WEAK* CONTINUOUS OPERATORS ON $\mathfrak{B}(\mathcal{H})$

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ABSTRACT. If Δ is a weak* continuous bounded linear operator on $\mathfrak{B}(\mathcal{H})$ that fixes the ideal of compact operators $\mathcal{K}$ and Δ_0 and δ are the induced maps on $\mathcal{K}$ and $\mathfrak{B}(\mathcal{H})/\mathcal{K}$ then it is shown that Δ has closed range, has dense range, is bounded below, or is onto if and only if both Δ_0 and δ have the same property. These results are then applied to the operator $X \to AXB$.

1. Introduction. We show that if Δ is a weak* continuous bounded linear operator on $\mathfrak{B}(\mathcal{H})$ that fixes the ideal of compact operators $\mathcal{K}$ and, Δ_0 and δ are the induced maps on $\mathcal{K}$ and $\mathfrak{B}(\mathcal{H})/\mathcal{K}$ then Δ has dense range, closed range, is bounded below, or is onto if and only if both Δ_0 and δ have the same property. These results are then applied to the operator $X \to AXB$.

2. Notation. Let $\mathcal{H}$ denote a separable infinite dimensional Hilbert space, $\mathcal{K}$ the ideal of compact operators on $\mathcal{H}$ and $\mathcal{C}(\mathcal{H})$ the Calkin algebra $\mathfrak{B}(\mathcal{H})/\mathcal{K}$. Let Δ: $\mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ denote a weak* continuous bounded linear operator such that $\Delta(\mathcal{K}) \subseteq \mathcal{K}$. Also let Δ_0 denote the restriction of Δ to $\mathcal{K}$ and let δ: $\mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ be the operator defined by $\delta(\Pi(X)) = \Pi(\Delta(X))$ for $X \in \mathfrak{B}(\mathcal{H})$ where Π: $\mathfrak{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is the natural projection.

Let $\mathfrak{T}$ be the ideal of trace-class operators with norm equal to the sum of the eigenvalues of $(T^*T)^{1/2}$. As Banach spaces, $\mathfrak{T}$ can be identified with the conjugate space of the ideal $\mathcal{K}$ by means of the linear isometry $T \to \mathfrak{f}_T$ where $\mathfrak{f}_T(K) = \operatorname{tr}(KT)$ for $K \in \mathcal{K}$. Moreover, $\mathfrak{B}(\mathcal{H})$ is the conjugate space of $\mathfrak{T}$. The weak* (or ultraweakly) continuous linear functionals on $\mathfrak{B}(\mathcal{H})$ are those of the form $\mathfrak{f}_T$ for some $T \in \mathfrak{T}$. The isometry $T \to \mathfrak{f}_T$ is an embedding of $\mathfrak{T}$ into $\mathfrak{B}(\mathcal{H})^*$ and $\mathfrak{B}(\mathcal{H})^*$ is the direct sum $\mathfrak{T} \oplus \mathcal{K}^0$ where $\mathcal{K}^0$ consists of those bounded linear functionals on $\mathfrak{B}(\mathcal{H})$ that annihilate the compact operators. Thus each $\mathfrak{f} \in \mathfrak{B}(\mathcal{H})^*$ has the form $\mathfrak{f} = \mathfrak{f}_T + \mathfrak{f}_0$ where $\mathfrak{f}_T$ is induced by a trace-class operator T and $\mathfrak{f}_0 \in \mathcal{K}^0$ (see [2]).

For $\mathfrak{f}_0 \in \mathcal{K}^0$ define the linear functional $\tilde{\mathfrak{f}}_0$ on $\mathcal{C}(\mathcal{H})$ by $\tilde{\mathfrak{f}}_0(\Pi(X)) = \mathfrak{f}_0(X)$ for $X \in \mathfrak{B}(\mathcal{H})$. Then $\mathfrak{f}_0 \to \tilde{\mathfrak{f}}_0$ is a linear isometry from $\mathcal{K}^0$ onto $\mathcal{C}(\mathcal{H})^*$.

Since Δ_0: $\mathcal{K} \to \mathcal{K}$, then Δ_0^*: $\mathfrak{T} \to \mathfrak{T}$ and Δ_0^{**}: $\mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$. Since Δ is assumed to be weak* continuous then $\Delta_0^{**} = \Delta$.

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[1] *Editor's Note*: Professor Weber passed away on March 12, 1980. His thesis advisor Professor James P. Williams, in spite of being seriously ill, took time to revise the paper on behalf of Professor Weber. Requests for reprints should be sent to Gary L. Mullen, a departmental colleague of Professor Weber.

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REMARK. The set of operators which satisfy the conditions imposed on Δ include operators of the form $X \to AX - XB$ which have recently been investigated by Fialkow [3], [4], and [5] and more generally those of the form $X \to A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n$ where $A_i, B_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, \ldots, n$. In fact the set of operators which satisfy these conditions is exactly the set of operators on $\mathcal{B}(\mathcal{H})$ of the form $\Delta = \Delta_0^{**}$ where Δ_0 is a bounded operator on $\mathcal{K}$.

For a bounded operator T on a Banach space, let $\sigma(T)$, $\sigma_d(T) = \{\lambda: T - \lambda$ is not onto$\}$, and $\sigma_\pi(T) = \{\lambda: T - \lambda$ is not bounded below$\}$ denote the spectrum, defect spectrum, and approximate point spectrum of T respectively. Also let $\mathcal{R}(T)$ denote the range of T and $\mathcal{N}(T)$ the null space of T respectively. Unless otherwise stated, the topology of $\mathcal{B}(\mathcal{H})$ is the norm topology.

For an operator $A \in \mathcal{B}(\mathcal{H})$, let Δ_A denote the derivation generated by A defined by $\Delta_A(X) = AX - XA$ for $X \in \mathcal{B}(\mathcal{H})$.

3. We begin with a basic relationship between Δ and $\Delta_0 \oplus \delta$.

LEMMA 1. *If Φ is the bounded linear operator defined on $\mathcal{B}(\mathcal{H})^*$ by $\Phi(\mathfrak{f}_T + \mathfrak{f}_0) = \mathfrak{f}_T + \tilde{\mathfrak{f}}_0$, then Φ is an isometry onto $\mathcal{K}^* + \mathcal{C}(\mathcal{H})^*$ and $(\Delta_0 \oplus \delta)^*\Phi = \Phi\Delta^*$.*

PROOF. Φ is the direct sum of two isometries and therefore is an isometry onto its range, which is $\mathcal{K}^* \oplus \mathcal{C}(\mathcal{H})^*$ by earlier remarks.

As first noticed by J. P. Williams [12], the annihilator of the range of a derivation on $\mathcal{B}(\mathcal{H})$ splits into the direct sum of its parts in $\mathcal{K}^0$ and $\mathcal{T}$. That this is true for the more general operator Δ is a consequence of Lemma 1.

LEMMA 2. $\mathcal{R}(\Delta)^0 = \mathcal{N}(\Delta_0^*) \oplus \mathcal{R}(\Delta)^0 \cap \mathcal{K}^0$.

PROOF. By Lemma 1,

$$\begin{aligned}\mathcal{R}(\Delta)^0 &= \mathcal{N}(\Delta^*) = \Phi^{-1}(\mathcal{N}(\Delta_0 \oplus \delta^*))\\ &= \Phi^{-1}(\mathcal{N}(\Delta_0^*) \oplus \mathcal{N}(\delta^*)) = \mathcal{N}(\Delta_0^*) \oplus \Phi^{-1}(\mathcal{N}(\delta^*)).\end{aligned}$$

If $\mathfrak{f} = \mathfrak{f}_0 + \mathfrak{f}_T \in \mathcal{R}(\Delta)^0 = \mathcal{N}(\Delta^*)$ then $0 = \Phi\Delta^*(\mathfrak{f}) = \mathfrak{f}_{\Delta_0^*(T)} \oplus \widetilde{\Delta^*\mathfrak{f}_0}$. Hence $\widetilde{\Delta^*(\mathfrak{f}_0)} = 0$ and so both $\mathfrak{f}_0$ and $\mathfrak{f}_T = \mathfrak{f} - \mathfrak{f}_0$ belong to $\mathcal{R}(\Delta)^0$.

We are now prepared to consider several properties of the range of Δ. We first consider conditions on Δ in order for the range to be dense in the weak* topology.

THEOREM 1. *The following are equivalent*:

(a) $\mathcal{R}(\Delta)$ *is weak* dense in* $\mathcal{B}(\mathcal{H})$.

(b) $\mathcal{R}(\Delta_0)$ *is dense in* $\mathcal{K}$.

(c) $\mathcal{K} \subseteq \mathcal{R}(\Delta)^-$.

(d) $\mathcal{N}(\Delta_0^*) = \{0\}$.

PROOF. The equivalence of (a) and (d) follows from the general duality fact that Δ_0^* is one-to-one if and only if $\mathcal{R}(\Delta_0^{**}) = \mathcal{R}(\Delta)$ is weak* dense (see [**10**, p. 94]). The equivalence of (b) and (d) follows from the fact that Δ_0^* is one-to-one if and only if $\mathcal{R}(\Delta_0)$ is dense. If $\mathcal{K} \subseteq \mathcal{R}(\Delta)^-$, then $\mathcal{R}(\Delta)^0 \subseteq \mathcal{K}^0$ and therefore by Lemma 2, $\mathcal{N}(\Delta_0^*) = \{0\}$. Hence (c) implies (d). That (b) implies (c) is obvious.

We now consider density in the norm topology.

THEOREM 2. *$\mathcal{R}(\Delta)$ is dense if and only if both $\mathcal{R}(\Delta_0)$ and $\mathcal{R}(\delta)$ are dense.*

PROOF. $\mathcal{R}(\Delta)$ is dense if and only if Δ^* is one-to-one. By Lemma 1, this is equivalent to $(\Delta_0 \oplus \delta)^*$ being one-to-one and hence to $\mathcal{R}(\Delta_0 \oplus \delta)$ being dense. However, $\mathcal{R}(\Delta_0 \oplus \delta)$ is dense if and only if $\mathcal{R}(\Delta_0)$ and $\mathcal{R}(\delta)$ are both dense.

We now consider conditions under which the range of Δ is closed.

Recall that for an operator T on a Banach space, $\mathcal{R}(T)$ is closed if and only if $\mathcal{R}(T^*)$ is closed (see [**10**, p. 96]).

THEOREM 3. *The following are equivalent.*

(a) *$\mathcal{R}(\Delta)$ is closed.*

(b) *$\mathcal{R}(\Delta_0)$ is closed.*

(c) *$\mathcal{R}(\Delta_0)$ and $\mathcal{R}(\delta)$ are both closed.*

PROOF. $\mathcal{R}(\Delta_0)$ is closed if and only if $\mathcal{R}(\Delta_0^*)$ is closed and hence if and only if $\mathcal{R}(\Delta) = \mathcal{R}(\Delta_0^{**})$ is closed. Therefore (a) and (b) are equivalent. Also $\mathcal{R}(\Delta)$ is closed if and only if $\mathcal{R}(\Delta^*)$ is closed and therefore, by Lemma 1, if and only if $\mathcal{R}(\Delta_0^*)$ and $\mathcal{R}(\delta^*)$, and therefore $\mathcal{R}(\Delta_0)$ and $\mathcal{R}(\delta)$ are both closed.

This theorem (as well as the next two) has an immediate corollary which displays an interesting fact concerning the relationship of operators on $\mathcal{B}(\mathcal{H})$ with the Calkin algebra.

COROLLARY. *If $\mathcal{R}(\Delta_0)$ is closed then $\mathcal{R}(\delta)$ is closed.*

The converse is false as can be seen by considering left multiplication by a one-one compact operator.

4. We now consider the relationship between the spectra of Δ, Δ_0, and δ. Recall that for a bounded operator T on a Banach space, $\sigma_d(T) = \sigma_\pi(T^*)$ and $\sigma_\pi(T) = \sigma_d(T^*)$ (see [**10**, pp. 94–97]).

THEOREM 4. *The following are equivalent*:

(a) *Δ is onto.*

(b) *Δ_0 is onto.*

(c) *Δ_0 and δ are both onto.*

(d) $\mathcal{K} \subseteq \mathcal{R}(\Delta)$.

PROOF. Since $\sigma_d(\Delta_0) = \sigma_\pi(\Delta_0^*) = \sigma_d(\Delta_0^{**}) = \sigma_d(\Delta)$ then (a) and (b) are equivalent. Also by Lemma 1, Δ^* is bounded below if and only if $(\Delta_0 \oplus \delta)^*$ is bounded below and hence if and only if Δ_0^* and δ^* are both bounded below. However, this is equivalent to Δ_0 and δ both being onto. Therefore (a) and (c) are equivalent. To show that (d) implies (a), assume that $\mathcal{K} \subset \mathcal{R}(\Delta)$. Then for $A \in \mathcal{B}(\mathcal{H})$ and Δ_A the derivation generated by A, $\mathcal{R}(\Delta_A|_{\mathcal{K}}) \subseteq \mathcal{R}(\Delta)$ and since $(\Delta_A|_{\mathcal{K}})^{**} = \Delta_A$, then $\mathcal{R}(\Delta_A) \subseteq \mathcal{R}(\Delta)$ (see [**8**, Corollary 1.2]). Therefore, since every operator is the sum of two commutators (Halmos [**7**], in fact there are isometries U and V with $\mathcal{R}(\Delta_U) + \mathcal{R}(\Delta_V) = \mathcal{B}(\mathcal{H})$, Weber [**11**]) Δ is onto.

COROLLARY. $\sigma_d(\delta) \subset \sigma_d(\Delta_0) = \sigma_d(\Delta)$.

PROOF. If Δ_0 is onto then by the theorem both Δ_0 and δ are onto and since the relationship between Δ, Δ_0, and δ is translation invariant the proof follows.

We have a similar result for the approximate point spectrum. The proof is similar to that of Theorem 4.

THEOREM 5. *The following are equivalent*:

(a) Δ *is bounded below.*

(b) Δ_0 *is bounded below.*

(c) Δ_0 *and* δ *are both bounded below.*

COROLLARY 1. $\sigma_\pi(\delta) \subseteq \sigma_\pi(\Delta_\sigma) = \sigma_\pi(\Delta)$.

By combining this corollary and the corollary to Theorem 4 we have

COROLLARY 2. $\sigma(\delta) \subseteq \sigma(\Delta_0) = \sigma(\Delta)$.

REMARK 1. Corollary 2 may also be proven directly using the facts that $\sigma(T) = \sigma(T^*)$ and that if Δ has an inverse Δ' then δ has an inverse $\delta'(\pi(X)) = \pi(\Delta'(X))$. Or better, using the isomorphism Φ, it is clear that $\sigma(\Delta) = \sigma(\Delta^*) = \sigma(\Delta_0^*) \cup \sigma(\delta^*) = \sigma(\Delta_0) \cup \sigma(\delta)$.

REMARK 2. Fialkow [3, p. 155] has shown by using the operator $X \to AX - XB$ that strict containment is possible in the corollaries to Theorems 4 and 5.

5. Examples. Several of the preceding results are generalizations of various properties of the operator $X \to AX - XB$ on $\mathfrak{B}(\mathcal{H})$ which were observed by L. A. Fialkow ([3], [4], and [5]). We will apply these results to the operator $X \to AXB$.

For A, $B \in \mathfrak{B}(\mathcal{H})$, let Γ: $\mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be the bounded linear operator defined by $\Gamma(X) = AXB$. Also let Γ_0 be the restrction of Γ to $\mathcal{K}$ and let γ be the operator on $\mathcal{C}(\mathcal{H})$ defined by $\gamma(\pi(X)) = \pi(\Gamma(X))$ for $X \in \mathfrak{B}(\mathcal{H})$. Lumer and Rosenblum [9] proved that $\sigma(\Gamma) = \sigma(A)\sigma(B) = \{ab\colon a \in \sigma(A) \text{ and } b \in \sigma(B)\}$ and, Davis and Rosenthal [1] proved that $\sigma_\pi(\Gamma) \subseteq \sigma_\pi(A)\sigma_d(B)$ and $\sigma_d(\Gamma) \subseteq \sigma_d(A)\sigma_\pi(B)$. The following lemma provides some information concerning the opposite inclusions.

For $x, y \in \mathcal{H}$, let $x \otimes y$ denote the rank one operator defined by $(x \otimes y)(z) = (z, y)x$ in which case $\|x \otimes y\| = \|x\|\,\|y\|$.

LEMMA 3. (a) *If* $0 \in \sigma_\pi(A) \cup \sigma_d(B)$, *then* $0 \in \sigma_\pi(\Gamma)$.

(b) *If* $0 \in \sigma_d(A) \cup \sigma_\pi(B)$, *then* $0 \in \sigma_d(\Gamma)$.

PROOF. If $0 \in \sigma_\pi(A)$ then there exists a sequence of unit vectors $\{x_n\}$ such that $\|Ax_n\| \to 0$. Therefore $\|A(x_n \otimes x_n)B\| = \|(Ax_n) \otimes \overline{(B^*x_n)}\| = \|Ax_n\|\,\|B^*x_n\| \to 0$ and Γ is not bounded below. If $0 \in \sigma_d(B) = \overline{\sigma_\pi(B^*)}$, then by choosing the unit vectors such that $\|B^*x_n\| \to 0$ it again follows that Γ is not bounded below. To establish (b), let $0 \in \sigma_d(A) \cup \sigma_\pi(B)$, then either A is not right invertible or B is not left invertible. In either case $AXB \neq I$ for any $X \in \mathfrak{B}(\mathcal{H})$ and hence Γ is not onto.

Since $\Gamma_0(K) = AKB$ for $K \in \mathcal{K}$, then Γ_0^*: $\mathcal{T} \to \mathcal{T}$ is the operator defined by $\Gamma_0^*(T) = BTA$ for all $T \in \mathcal{T}$. We next establish a spectral condition under which Γ_0^* is one-to-one.

In the following we let $\sigma_p(T)$ and $\sigma_c(T) = \{\lambda\colon \mathfrak{R}(T - \lambda) \text{ is not dense}\}$ be the point and compression spectrum respectively of an operator T on a Banach space.

LEMMA 4. *Γ_0^* is one-to-one if and only if $0 \notin \sigma_c(A) \cup \sigma_p(B)$.*

PROOF. Assume $0 \notin \sigma_p(B)$ and A has dense range. If T is a nonzero trace-class operator, choose $x \in \mathcal{H}$ such that $Tx = y \neq 0$. Also let $\{z_n\}$ be a sequence in $\mathcal{H}$ such that $Az_n \to x$. Then $TAz_n \to y$ and $BTAz_n \to By \neq 0$. Therefore $BTA \neq 0$ and $N(\Gamma_0^*) = \{0\}$. Conversely if $y \in \mathcal{H}$ is not the zero vector and either $By = 0$ or $y \perp \mathfrak{R}(A)^-$ then $B(y \otimes y)A = (By) \otimes (A^*y) = 0$. Hence $N(\Gamma_0^*) \neq \{0\}$.

Next we consider the density of the range of Γ.

Recall that the weakly (weak operator topology) continuous linear functionals on $\mathfrak{B}(\mathcal{H})$ are those of the form $\mathfrak{f}_F$ where F is a finite rank operator on $\mathcal{H}$.

THEOREM 6. *The following are equivalent*:

(a) *$\mathfrak{R}(\Gamma)$ is weak* dense in $\mathfrak{B}(\mathcal{H})$.*

(b) *$\mathfrak{R}(\Gamma)$ is weakly dense in $\mathfrak{B}(\mathcal{H})$.*

(c) *$\mathfrak{R}(\Gamma_0)$ is dense in $\mathcal{H}$.*

(d) *$\mathcal{H} \subseteq \mathfrak{R}(\Gamma)^-$.*

(e) *$0 \notin \sigma_c(A) \cup \sigma_p(B)$.*

PROOF. In view of Theorem 1 and Lemma 4, it is sufficient to show that (b) implies (a). If $\mathfrak{R}(\Gamma)$ is not weak* dense then $0 \in \sigma_c(A) \cup \sigma_p(B)$ and, as in the proof of Lemma 4, there exists a nonzero finite rank operator F such that $BFA = 0$. Therefore $\mathfrak{f}_F(AXB) = \operatorname{tr}(AXBF) = \operatorname{tr}(XBFA) = 0$ for all $X \in \mathfrak{B}(\mathcal{H})$ and $\mathfrak{R}(\Gamma)$ is not weakly dense.

We now turn to the operator γ.

Let $\sigma_e(t)$, $\sigma_{re}(T)$, and $\sigma_{le}(T)$ denote the essential spectrum, right essential spectrum, and left essential spectrum respectively of an operator $T \in \mathfrak{B}(\mathcal{H})$. Also for $A, B \in \mathfrak{B}(\mathcal{H})$, let $\mathfrak{A}$ and $\mathfrak{B}$ denote the operators on the Calkin algebra defined by $\mathfrak{A}(\tilde{X}) = \tilde{A}\tilde{X}$ and $\mathfrak{B}(\tilde{X}) = \tilde{X}\tilde{B}$ for $X \in \mathfrak{B}(\mathcal{H})$ and $\tilde{X} = \pi(X)$. Then, as observed by Fialkow [3], $\sigma_\pi(\mathfrak{A}) = \sigma_{le}(A)$, $\sigma_\pi(\mathfrak{B}) = \sigma_{re}(B)$, $\sigma_d(\mathfrak{A}) = \sigma_{re}(A)$, and $\sigma_d(\mathfrak{B}) = \sigma_{le}(B)$.

LEMMA 5. (a) $\sigma(\gamma) \subseteq \sigma_e(A)\sigma_e(B)$.

(b) $\sigma_\pi(\gamma) \subseteq \sigma_{le}(A)\sigma_{re}(B)$.

(c) $\sigma_d(\gamma) \subseteq \sigma_{re}(A)\sigma_{le}(B)$.

PROOF. Part (a) is due to Lumer and Rosenblum [**9**, Theorem 5]. Also by a result of Davis and Rosenthal [**1**, Theorem 2] $\sigma_\pi(\gamma) \subseteq \sigma_\pi(\mathfrak{A})\sigma_\pi(\mathfrak{B}) = \sigma_{le}(A)\sigma_{re}(B)$ and $\sigma_d(\gamma) \subseteq \sigma_d(\mathfrak{A})\sigma_d(\mathfrak{B}) = \sigma_{re}(A)\sigma_{le}(B)$.

Recall that if $0 \in \sigma_{le}(A)(0 \in \sigma_{re}(A))$ then there exists an infinite dimensional projection P such that $AP(PA)$ is compact (see [**6**, Theorem 4.1]).

LEMMA 6. (a) *If* $0 \in \sigma_e(A) \cup \sigma_e(B)$ *then* $0 \in \sigma(\gamma)$.

(b) *If* $0 \in \sigma_{le}(A) \cup \sigma_{re}(B)$ *then* $0 \in \sigma_p(\gamma)$.

(c) *If* $0 \in \sigma_{re}(A) \cup \sigma_{le}(B)$ *then* $0 \in \sigma_d(\gamma)$.

PROOF. If $0 \in \sigma_e(A) \cup \sigma_e(B)$ then there exists an infinite dimensional projection P such that one of AP, PA, BP, or PB is compact. If AP or PB is compact, then

$\gamma(\pi(P)) = \pi(APB) = \tilde{0}$ and γ is not one-to-one. Also if PA or BP is compact and if $\gamma(\pi(X)) = \pi(AXB) = \tilde{I}$ then $\tilde{P} = \tilde{P}\tilde{I} = \pi(PAXB) = \pi(AXBP) = \tilde{0}$ which again contradicts the fact that P is infinite dimensional. Hence γ is not onto. Parts (b) and (c) follow by applying the above arguments to the various cases.

Fialkow [5] has shown that for the operator Δ: $X \to AX - XB$, $\mathfrak{R}(\delta)$ is dense if and only if δ is onto. This is also true for the operator Γ.

LEMMA 7. *$\mathfrak{R}(\gamma)$ is dense if and only if γ is onto.*

PROOF. Assume γ is not onto and hence $0 \in \sigma_d(\gamma) \subseteq \sigma_{re}(A)\sigma_{le}(B)$. If $0 \in \sigma_{re}(A)$ then let P be an infinite dimensional projection such that PA is compact. If $\mathfrak{R}(\gamma)$ is dense then there exists a sequence $\{\tilde{X}_n\}$ such that $\tilde{A}\tilde{X}_n\tilde{B} \to \tilde{I}$. Hence $\tilde{0} = \tilde{P}\tilde{A}\tilde{X}_n\tilde{B} \to \tilde{P}$ which contradicts the fact that P is infinite dimensional. A similar argument can be employed if we assume that $0 \in \sigma_{le}(B)$.

With regard to norm density we have the following equivalences.

THEOREM 7. *The following are equivalent*:

(a) *$\mathfrak{R}(\Gamma)$ is dense in $\mathfrak{B}(\mathfrak{H})$.*

(b) *γ is onto and $\mathfrak{R}(\Gamma_0)$ is dense in $\mathfrak{H}$.*

(c) *$0 \notin \sigma_{re}(A) \cup \sigma_c(A) \cup \sigma_{le}(B) \cup \sigma_p(B)$.*

PROOF. The equivalence of (a) and (b) follows directly from Theorem 2 and Lemma 7. Also γ is onto if and only if $0 \notin \sigma_{re}(A) \cup \sigma_{le}(B)$ and $\mathfrak{R}(\Gamma_0)$ is dense if and only if $N(\Gamma_0^*) = \{0\}$ and hence, by Lemma 4, if and only if $0 \notin \sigma_c(A) \cup \sigma_p(B)$.

COROLLARY. *$\mathfrak{R}(\Gamma)$ is dense if and only if Γ is onto.*

PROOF. Since

$$\sigma_d(A) \cup \sigma_\pi(B) = \overline{\sigma_\pi(A^*)} \cup \sigma_\pi(B) = \overline{\sigma_{le}(A^*)} \cup \overline{\sigma_p(A^*)} \cup \sigma_{le}(B) \cup \sigma_p(B)$$
$$= \sigma_{re}(A) \cup \sigma_c(A) \cup \sigma_{le}(B) \cup \sigma_p(B),$$

the proof follows from Theorem 7.

We now consider other conditions under which Γ is onto.

THEOREM 8. *The following are equivalent*:

(a) *Γ is onto.*

(b) *$\mathfrak{R}(\Gamma)$ is dense in $\mathfrak{B}(\mathfrak{H})$.*

(c) *Γ_0 is onto.*

(d) *γ and Γ_0 are both onto.*

(e) *$\mathfrak{R}(\Gamma)$ contains all rank-one operators.*

(f) *$0 \notin \sigma_d(A)\sigma_\pi(B)$.*

PROOF. In view of Theorem 4 and Lemma 3, it remains only to show that (e) implies (a). Therefore assume that Γ is not onto in which case $0 \in \sigma_d(A)\sigma_\pi(B)$. If $0 \in \sigma_\pi(B) \subseteq \sigma_{le}(B) \cup \sigma_p(B)$ and $\mathfrak{R}(\Gamma)$ contains all rank-one operators then $0 \in \sigma_{le}(B)$. Let $\{\mathfrak{f}_n\}$ be an orthonormal sequence such that $\Sigma\|B\mathfrak{f}_n\|^{1/2} < \infty$ (see [6]).

Then

$$\sum |((AXB)\mathfrak{f}_n, \mathfrak{f}_n)|^{1/2} \leq \sum (\|AXB\mathfrak{f}_n\| \, \|\mathfrak{f}_n\|)^{1/2}$$
$$\leq \sum \|AX\|^{1/2}\|B\mathfrak{f}_n\|^{1/2} \leq \|AX\|^{1/2}\sum \|B\mathfrak{f}_n\|^{1/2} < \infty$$

for all $X \in \mathfrak{B}(\mathcal{H})$. Since $\mathfrak{R}(\Gamma)$ contains all rank-one operators, then for all $\mathfrak{f}\mathcal{H}$, $\Sigma|((\mathfrak{f} \otimes \mathfrak{f})\mathfrak{f}_n, \mathfrak{f}_n)|^{1/2} < \infty$. However

$$\sum |((\mathfrak{f} \otimes \mathfrak{f})\mathfrak{f}_n, \mathfrak{f}_n)|^{1/2} = \sum |(\mathfrak{f}, \mathfrak{f}_n)\overline{(\mathfrak{f}_n, f)}\,|^{1/2} = \sum |(\mathfrak{f}_n, \mathfrak{f})|.$$

Hence if we choose $\mathfrak{f}$ such that $\{|(\mathfrak{f}_n, \mathfrak{f})|\}$ is not summable, we have a contradiction and therefore $0 \notin \sigma_\pi(B)$. If we assume $0 \in \sigma_d(A) = \overline{\sigma_\pi(A^*)}$ we can again reach a contradiction by choosing the orthonormal sequence $\{\mathfrak{f}_n\}$ such that $\Sigma\|A^*\mathfrak{f}_n\|^{1/2} < \infty$.

Finally, we give conditions under which Γ is bounded below.

THEOREM 9. *The following are equivalent*:

(a) Γ *is bounded below.*

(b) Γ_0 *is bounded below.*

(c) γ *and* Γ_0 *are both bounded below.*

(d) *There exists an* M *such that* $\|AXB\| \geq M\|X\|$ *for all rank-one operators* $X \in \mathfrak{B}(\mathcal{H})$.

(e) $0 \notin \sigma_\pi(A)\sigma_d(B)$.

PROOF. In view of Theorem 5, and Lemma 3 it only remains to show that (d) implies (a). Therefore assume Γ is not bounded below and hence $0 \in \sigma_\pi(A)\sigma_d(B)$. If $0 \in \sigma_\pi(A)$ then for $\alpha > 0$ choose the unit vector x such that $\|Ax\| < \varepsilon/\|B^*\|$. Then

$$\|A(x \otimes x)By\| = \|(Ax) \otimes (B^*x)y\| = \|(y, B^*x)Ax\|$$
$$\leq \|y\| \, \|B^*x\| \, \|Ax\| \leq \|y\| \, \|B^*\| \frac{\alpha}{\|B^*\|}$$
$$\leq \|y\|\alpha \quad \text{for all } y \in \mathcal{H}.$$

Therefore $\|A(x \otimes x)B\| \leq \alpha$ and Γ is not bounded below on rank-one operators. If $0 \in \sigma_d(B)$ then a similar argument can be applied to B^*.

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