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A NOTE ON $L(p, q)$ SPACES AND ORLICZ SPACES WITH MIXED NORMS

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ABSTRACT. Necessary and sufficient conditions are given for the embedding of $L(p, q)$ spaces and Orlicz spaces with mixed norms.

1. Introduction. Let $X(0, \infty)$, $Y(0, \infty)$, $Z((0, \infty) \times (0, \infty))$ be Banach function spaces (cf. [6]) of Lebesgue measurable functions on $(0, \infty)$ (respectively $(0, \infty) \times (0, \infty)$). Let M denote the class of real valued Lebesgue measurable functions on $(0, \infty) \times (0, \infty)$ and let $[X, Y] = \{k \in M: \|k\|_{[X,Y]} < \infty\}$, where

$$\|k\|_{[X,Y]} = \sup\left\{\int\int |k(x,y)f(y)g(x)|\, dx\, dy: \|f\|_X \leq 1, \|g\|_{Y'} \leq 1\right\}.$$

Therefore, if $k \in [X, Y]$ then $z_k(f) = \int k(x,y)f(y)\, dy$ defines a bounded linear operator $z_k: X \to Y$.

In the qualitative theory of integral equations (cf. [2]) it is important to determine necessary and sufficient conditions for a continuous embedding $Z \subseteq [X, Y]$ to hold. It is well known, and easy to see, that Z is continuously embedded in $[X, Y]$ if and only if $X \otimes_\pi Y'$ is continuously embedded in Z' (cf. [8]).

In [10] the following results are obtained for tensor products of $L(p, q)$ spaces and Orlicz spaces (the symbol $\subseteq$ will be used to denote a continuous embedding).

THEOREM A. *Let A, B and C be Young's functions, then the following statements are equivalent.*

(i) $\exists \theta > 0 \ni A^{-1}(t)B^{-1}(s) \leq \theta C^{-1}(ts), \forall t, s > 0$,

(ii) $L_A \otimes_\pi L_B \subseteq L_C$,

(iii) $L_A \otimes_\pi M(L_B) \subseteq M(L_C)$.

THEOREM B. *Let $1 < p < \infty$, $1 \leq q_i \leq \infty$, $i = 1, 2, 3$. Then,*

$$L(p, q_1) \otimes_\pi L(p, q_2) \subseteq L(p, q_3)$$

if and only if the following conditions are satisfied.

(i) $\max\{q_1, q_2\} \leq q_3$,

(ii) $1/p + 1/q_3 \leq 1/q_1 + 1/q_2$.

In this paper we consider $L(p, q)$ spaces and Orlicz spaces with mixed norms. Let $X(Y)$ denote the space defined as follows.

$$X(Y) = \{f \in M: \|f\|_{X(Y)} = \| \|f(x, \cdot)\|_Y \|_X < \infty\}.$$

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It is clear that $X \otimes_\pi Y \subseteq X(Y)$ and therefore we have

(1.1) $$X(Y) \subseteq Z \Rightarrow X \otimes_\pi Y \subseteq Z.$$

We prove the following

THEOREM 1. *Let A, B and C be Young's functions, then the following statements are equivalent.*

(i) $\exists \theta > 0 \ni A^{-1}(t)B^{-1}(s) \leqslant \theta C^{-1}(ts)$, $\forall t, s > 0$,

(ii) $L_A(L_B) \subseteq L_C$,

(iii) $L_A(M(L_B)) \subseteq M(L_C)$.

THEOREM 2. *Let $1 < p < \infty$, $1 \leqslant q_i \leqslant \infty$, $i = 1, 2, 3$. Then, $L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3)$ if and only if the following conditions are satisfied.*

(i) $\max\{q_1, q_2\} \leqslant q_3$,

(ii) $q_1 \leqslant p \leqslant q_3$.

The sufficiency of the conditions of Theorem 2 was obtained by Walsh **[11]**.

The reader will be assumed familiar with the theory of $L(p, q)$ spaces and Orlicz spaces (cf. **[10]**, **[5]**). We shall follow the notation of **[10]**. Note, however, that the Marcinkiewicz M_A spaces of **[10]** are denoted by $M(L_A)$ in the present work.

2. Proof of Theorem 1. We begin by recalling a construction of A. P. Calderón **[1]** which plays an important role in the theory of tensor products of function spaces (cf. **[8]**).

Let (Ω, μ) be a measure space, $X(\Omega)$ a Banach function space, $\Sigma(X)$ its unit ball, and A a Young's function. For each pair $(X(\Omega), A)$ define

$$A^{-1}(X)(\Omega) = \{f \in M(\Omega) \colon \exists \lambda > 0 \text{ and } g \in \Sigma(X) \ni |f(x)| \leqslant \lambda A^{-1}(|g(x)|)\}$$

equipped with its natural norm $A^{-1}(X)$ becomes a Banach function space. Observe that $A^{-1}(L^1) = L_A$ and $A^{-1}(L(1, \infty)) \cong M(L_A)$ if A satisfies the ∇_2 condition.

The following result was obtained in **[8]**.

(2.1) LEMMA. *Let X, Y and Z be Banach function spaces (as in* §1*), and A, B and C Young's functions. Moreover, assume that condition* (i) *of Theorem* A *is satisfied, then,*

(i) $X \otimes_\pi Y \subseteq Z \Rightarrow A^{-1}(X) \otimes_\pi B^{-1}(Y) \subseteq C^{-1}(Z)$,

(ii) $X(Y) \subseteq Z \Rightarrow A^{-1}(X)(B^{-1}(Y)) \subseteq C^{-1}(Z)$.

Now observe that $L^1(L^1) = L^1$ and use (2.1) to obtain the equivalence of (i) and (ii). Moreover, since $L^1(L(1, \infty)) \subseteq L(1, \infty)$, (2.1) implies the equivalence of (i) and (iii), if we put the additional condition that B and C satisfy the ∇_2 condition. To remove this condition we use the following.

(2.2) LEMMA (CF. **[9]**). *Let X, Y and Z be rearrangement invariant spaces (as in* §1*). Suppose that there exists a constant $M > 0$ such that $\forall u \in L^1(0, \infty)$ we have*

$$\|\phi_{Y'}(|u|)\|_{X'} \leqslant M\phi_{Z'}(\|u\|_1),$$

where $\phi_{Y'}$ (respectively $\phi_{Z'}$) denotes the fundamental function of Y' (respectively Z') (cf. **[12]**). *Then,*

$$X(M(Y)) \subseteq M(Z).$$

In order to prove the implication (i) $\Rightarrow$ (iii) we apply (2.2). In fact a simple computation shows that (i) implies $\phi_{L_{\bar{B}}}(s)/\phi_{L_{\bar{C}}}(t) \leqslant \theta \bar{A}^{-1}(s/t)$, $\forall t, s > 0$. Therefore,

$$\bar{A}\left(\phi_{L_{\bar{B}}}(s)/\theta\phi_{L_{\bar{C}}}(t)\right) \leqslant s/t, \qquad \forall t, s > 0.$$

Let $u \in L^1(0, \infty)$, then by the above inequality, we have

$$A\left(\phi_{L_{\bar{B}}}(|u(x)|)/\theta\phi_{L_{\bar{C}}}(\|u\|_1)\right) \leqslant |u(x)|/\|u\|_1 \quad \text{a.e.},$$

which readily implies that $\|\phi_{L_{\bar{B}}}(|u|)\|_{L_{\bar{A}}} \leqslant \theta\phi_{L_{\bar{C}}}(\|u\|_1)$. Therefore (iii) holds by (2.2).

The reverse implication (iii) $\Rightarrow$ (i) is trivial and follows from Theorem A and (1.1).

(2.3) REMARK. For Orlicz spaces defined on finite measure spaces the equivalence (i) $\Leftrightarrow$ (ii) was proved in **[4]**. We point out that similar results hold for Orlicz-Marcinkiewicz spaces defined by generalized Young's functions (cf. **[10]**, **[8]**).

3. Proof of Theorem 2. For the sake of completeness we prove that the conditions (i) and (ii) are sufficient. Let p' be defined by $1/p + 1/p' = 1$, and let $u \in L^1(0, \infty)$, then $\| \, |u|^{1/p'}\|_{p'} = \|u\|_1^{1/p'}$. Therefore, by (2.2), $L^p(L(p, \infty)) \subseteq L(p, \infty)$ (cf. **[11]**). Suppose now that (i) and (ii) hold, consider two cases: $q_2 \leqslant p$ or $q_2 > p$. In the first case we have

$$L(p, q_1)(L(p, q_2)) \subseteq L^p \subseteq L(p, q_3).$$

In the second case we obtain the desired result interpolating (by the complex method) between $L^p(L(p, \infty)) \subseteq L(p, \infty)$ and $L^p(L^p) = L^p$ (cf. **[5]**).

The necessity of (i) follows from (1.1) and Theorem B, while the necessity of (ii) will follow from (3.1) and (3.2) below.

We use some constructions of Cwikel [3].

(3.1) LEMMA. $L(p, q_1)\,(L(p, q_2)) \subseteq L(p, \infty) \Rightarrow q_1 \leqslant p$.

PROOF. Consider two cases: $q_2 = \infty$ or $q_2 < \infty$. In the first case we get $q_1 \leqslant p$ using (1.1) and Theorem B. Suppose now that $q_2 < \infty$ and $q_1 > p$. We shall construct $f \in L(p, q_1)\,(L(p, q_2))$ such that $f \notin L(p, \infty)$.

Let us choose $0 < \varepsilon < 1$ such that $p < \varepsilon q_1$, and define (cf. **[3]**)

$$f(x, y) = \chi_{(0, e(x))}(y) F(x), \qquad x, y \in (0, \infty),$$

where $e(x) = (x + 1)^{-1}$, $F(x) = \min\{1, [\log(x + 1)]^{-\varepsilon/p}\}$. Then,

$$\| \, \|f(x, \cdot)\|_{p, q_2}\|_{p, q_1}^{q_1} \sim \int_{e-1}^{\infty} (1 + x)^{-q_1/p}\left[\log(x + 1)\right]^{-\varepsilon q_1/p} x^{q_1/p} \, \frac{dx}{x} < \infty.$$

On the other hand,

$$\begin{aligned} \lambda_f(t) &= |\{(x, y) : 0 < y \leqslant e(x), F(x) > t\}| \\ &= \log(\lambda_F(t) + 1). \end{aligned}$$

Therefore, $f^*(t) = F(e^t - 1)$. Thus, for $t \geqslant 1$, $f^*(t) = t^{-\varepsilon/p}$ and

$$\|f\|_{p, \infty} = \sup_{t > 0}\left\{f^*(t) t^{1/p}\right\} \geqslant \sup_{t \geqslant 1}\left\{t^{-\varepsilon/p} t^{1/p}\right\} = \infty.$$

(3.2) LEMMA. *If* $L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3)$, *then* $p \leqslant q_3$.

PROOF. Assume without loss that $q_3 < \infty$. Suppose that $L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3)$ but to the contrary $q_3 < p$.

Let us choose $0 < \varepsilon < 1 - q_3/p$, and define f as in (3.1) with

$$e(x) = e^x, \quad F(x) = e^{-x/p}\chi_{(0,1)}(x) + e^{-x/p}x^{-1/p-\varepsilon/q_3}\chi_{[1,\infty)}(x).$$

Then, as shown by Cwikel [3], $\|f\|_{p,q_3} = \infty$. However,

$$\| \|f(x, \cdot)\|_{p,q_2}\|_{p,q_1}^{q_1} \sim \int_1^\infty x^{-\varepsilon q_1/q_3} \frac{dx}{x} < \infty,$$

a contradiction.

To complete the proof of Theorem 2 proceed as follows. Suppose $L(p, q_1)(L(p, q_2)) \subseteq L(p, q_3)$, then $L(p, q_1)(L(p, q_2)) \subseteq L(p, \infty)$. Now apply (3.1), (3.2), (1.1) and Theorem B to obtain the necessity of conditions (i)–(ii).

(3.3) REMARK. In our announcement [7], condition (ii) of Theorem 2 is incorrectly stated as $\max(q_1, q_2) \leqslant p \leqslant q_3$.

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