PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

CONTRACTIVE COMMUTANTS AND INVARIANT SUBSPACES

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ABSTRACT. Let T be a bounded operator on a Banach space $\mathfrak{X}$ and let K be a nonzero compact operator. In [1] and [4] it is shown that if λ is a complex number and if $TK = \lambda KT$, then T has a hyperinvariant subspace. In [1], S. Brown goes on to show that if $\mathfrak{X}$ is reflexive and if $TK = \lambda KT$ and $TB = \mu BT$ for some λ, μ with $|\lambda| \neq 1$ and $(1 - |\mu|)/(1 - |\lambda|) > 0$, then B has an invariant subspace. Below we extend both these results by showing that the entire class of operators satisfying the above conditions on B has an invariant subspace.

Let T be a bounded operator on a Banach space $\mathfrak{X}$ and let K be a nonzero compact operator. In [1] and [4] it is shown that if λ is a complex number and if $TK = \lambda KT$, then T has a hyperinvariant subspace. In [1], S. Brown goes on to show that if $\mathfrak{X}$ is reflexive and if $TK = \lambda KT$ and $TB = \mu BT$ for some λ, μ with $|\lambda| \neq 1$ and $(1 - |\mu|)/(1 - |\lambda|) \geqslant 0$, then B has an invariant subspace. In this note we extend both these results by showing that the entire class of operators satisfying the above conditions on B has an invariant subspace.

1. The contractive commutant. Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let $\mathcal{L}(\mathfrak{X})$ be the algebra of bounded linear operators on $\mathfrak{X}$. The commutant $\{T\}'$ of an operator T is the algebra of operators B that commute with T. A basic result is the elegant theorem of Lomonosov [5]; the statement below is the distillation by Pearcy and Shields [6].

THEOREM 1.1 (LOMONOSOV). *If $\mathcal{A}$ is a subalgebra of $\mathcal{L}(\mathfrak{X})$ with no nontrivial invariant subspaces, and if K is any nonzero compact operator, then there is an operator A in $\mathcal{A}$ such that 1 is an eigenvalue of AK.*

COROLLARY 1.2 (LOMONOSOV). *If $\{T\}'$ contains a nonzero compact operator and T is not a scalar multiple of the identity, then $\{T\}'$ has an invariant subspace.*

Let $\mathcal{C}_c(T) = \{B \in \mathcal{L}(\mathfrak{X})\colon TB = \lambda BT$ for some complex number λ with $|\lambda| \leqslant 1\}$. Notice that $\mathcal{C}_c(T)$ is not an algebra, since it fails to be closed under sums. Let $\{T\}'_c$ be the (nonclosed) algebra generated by $\mathcal{C}_c(T)$. We refer to $\{T\}'_c$ as the *contractive commutant* of T. Similarly, let $\mathcal{C}_{sc}(T) = \{B \in \mathcal{L}(\mathfrak{X})\colon TB = \lambda BT$ with $|\lambda| < 1\}$ and let $\{T\}'_{sc}$ be the algebra generated by $\mathcal{C}_{sc}(T)$; we call $\{T\}'_{sc}$ the *strictly contractive commutant* of T. A number of simple facts are listed below.

Received by the editors May 27, 1980 and, in revised form, April 3, 1981.
1980 *Mathematics Subject Classification*. Primary 47A15; Secondary 47B05.

0002-9939/81/0000-0566/$02.00

THEOREM 1.3. (i) *If A, B are in $\mathcal{C}_c(T)$ (resp. $\mathcal{C}_{sc}(T)$), and if $\mu \in \mathbf{C}$ then μA and AB are in $\mathcal{C}_c(T)$ (resp. $\mathcal{C}_{sc}(T)$).*

(ii) *If $A \in \mathcal{C}_c(T)$ and $B \in \mathcal{C}_{sc}(T)$ then BA and AB lie in $\mathcal{C}_{sc}(T)$.*

(iii) *If $A \in \mathcal{C}_c(T)$ (resp. $\mathcal{C}_{sc}(T)$) then $T^* \in \mathcal{C}_c(A^*)$ (resp. $\mathcal{C}_{sc}(A^*)$).*

(iv) *$\mathcal{C}_c(T)$ is closed in the weak operator topology.*

PROOF. (i), (ii), and (iii) are straightforward computations. To prove (iv) we suppose that $\{B_\alpha\}$ is a net of operators in $\mathcal{C}_c(T)$ and $B_\alpha \to B$ weakly. If λ_α is chosen so that $TB_\alpha = \lambda_\alpha B_\alpha T$, then $|\lambda_\alpha| \leq 1$ for all α and thus there is a convergent subnet of $\{\lambda_\alpha\}$; without loss of generality we assume that the entire net $\{\lambda_\alpha\}$ converges, say to λ. Then TB_α converges weakly to TB, $\lambda_\alpha B_\alpha T$ converges weakly to λBT, and the result follows.

LEMMA 1.4. *$\{T\}'_c$ (resp. $\{T\}'_{sc}$) consists precisely of finite sums, $\Sigma_{i=1}^n B_i$, where each B_i lies in $\mathcal{C}_c(T)$ (resp. $\mathcal{C}_{sc}(T)$).*

PROOF. $\{T\}'_c$ is the algebra generated by $\mathcal{C}_c(T)$, so clearly every finite sum of operators in $\mathcal{C}_c(T)$ belongs to $\{T\}'_c$. It is easy to check, using 1.3(i), that the collection of finite sums is an algebra, and thus that it is the same as $\{T\}'_c$. The statement for $\{T\}'_{sc}$ follows similarly.

COROLLARY 1.5. *If $A \in \{T\}'_c$ and $B \in \{T\}'_{sc}$, then AB and BA lie in $\{T\}'_{sc}$.*

PROOF. Use Lemma 1.4 and Theorem 1.3(iii).

The proof of the next result is a slight sharpening of the proof of Theorem 2 of **[1]**.

THEOREM 1.6. (i) *If $TB = \lambda BT$ for some complex number λ (not necessarily in the unit disk) then either $|\lambda| = 1$ or TB and BT are quasinilpotent.*

(ii) *If $TK = \lambda KT$ where K is compact then either λ is a root of unity or TK and KT are quasinilpotent.*

PROOF. (i) It is well known that the nonzero elements of $\sigma(TB)$ and $\sigma(BT)$ are the same **[3**, p. 63]. Thus it follows that $r(TB) = r(BT)$, where $r(X)$ denotes the spectral radius of X. Since $TB = \lambda BT$ we also have $r(TB) = |\lambda| r(BT)$ and thus $r(BT) = |\lambda| r(BT)$. Hence either $|\lambda| = 1$ or else $r(BT)$ (and therefore $r(TB)$) is 0.

(ii) Suppose $TK = \lambda KT$ and TK and KT are not quasinilpotent. By part (i), $|\lambda| = 1$. Let $0 \neq z \in \sigma(KT)$. Then $\lambda z \in \sigma(TK) = \sigma(KT)$. By induction, $\lambda^n z \in \sigma(KT)$ for all nonnegative integers n. However, KT is compact and its spectrum cannot contain an infinite set of numbers whose absolute values are bounded away from 0. Thus $\{\lambda^n z\}_{n=0}^\infty$ is a finite set and λ must be a root of unity.

2. Invariant subspaces. The contractive commutant contains the commutant; thus it is less likely that the former should have nontrivial invariant subspaces. The following example shows that $\{T\}'_c$ may indeed be transitive.

EXAMPLE 2.1. Let $\mathcal{K}$ be a two-dimensional Hilbert space and let T, K_1 and K_2 be defined by

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad K_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then K_1 and K_2 both lie in $\mathcal{C}_c(T)$, but it is easy to see that no subspace is invariant for T, K_1, and K_2. Notice that in this case, $\{T\}'_c = \mathcal{L}(\mathfrak{X})$.

For an infinite-dimensional example, let $\mathcal{H}$ be any Hilbert space and let $\mathfrak{X} = \mathcal{H} \oplus \mathcal{H}$. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{L}(\mathfrak{X})$. $\mathcal{C}_c(T)$ contains all operators of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}.$$

Thus $\{T\}'_c = \mathcal{L}(\mathfrak{X})$ in this case as well. Observe that if $\{T\}'_c$ were transitive and not dense in $\mathcal{L}(\mathfrak{X})$ we would have a solution to the transitive algebra problem.

Our central result shows that under certain conditions $\{T\}'_c$ and $\{T\}'_{sc}$ do have invariant subspaces.

Theorem 2.2. *Let T be a nonzero operator in $\mathcal{L}(\mathfrak{X})$. If $\{T\}'_{sc}$ (resp. $\{T\}'_c$) contains a nonzero compact operator, then $\{T\}'_c$ (resp. $\{T\}'_{sc}$) has a nontrivial invariant subspace.*

Proof. Let $\{T\}'_{sc}$ contain a nonzero compact operator K. Note that $\ker T$ is an invariant subspace for $\mathcal{C}_c(T)$ and hence for $\{T\}'_c$; we therefore assume that $\ker T = \{0\}$. Suppose that $\{T\}'_c$ is a transitive algebra. Theorem 1.1 guarantees the existence of an operator B in $\{T\}'_c$ and a nonzero vector x such that $BKx = x$. By Corollary 1.5, $BK \in \{T\}'_{sc}$ and thus there exist $B_1, \ldots, B_n \in \mathcal{C}_{sc}(T)$ such that $\sum_{i=1}^n B_i = BK$. Let $TB_i = \lambda_i B_i T$, where $|\lambda_i| < 1$ for each i. Then $TBK = \sum TB_i = (\sum \lambda_i B_i)T$ and inductively $T^m BK = (\sum_{i=1}^n \lambda_i^m B_i)T^m$ for each positive integer m. Hence $T^m x = T^m BKx = (\sum_{i=1}^n \lambda_i^m B_i)T^m x$. We have assumed that T has trivial kernel and thus $T^m x \neq 0$ for every m, and it follows that 1 lies in the point spectrum of $\sum_{i=1}^n \lambda_i^m B_i$ for every m. However, this would imply that $1 \leqslant \|\sum_{i=1}^n \lambda_i^m B_i\| \leqslant \sum_{i=1}^n |\lambda_i|^m \, \|B_i\|$ for all m, which is obviously impossible since $|\lambda_i| < 1$ for all i. Hence the assumption that $\{T\}'_c$ is transitive must be false.

The proof of the other part of the theorem is virtually identical and is omitted.

The corollary is a generalization of Theorem 3 of **[1]**.

Corollary 2.3. *Suppose that $\mathfrak{X}$ is reflexive and that $TK = \lambda KT$ for K a nonzero compact operator, T nonzero, and $|\lambda| \neq 1$. Let $\mathfrak{a}$ be the algebra generated by all operators B such that $TB = \mu BT$ for some complex number μ for which $(1 - |\mu|)/(1 - |\lambda|) \geqslant 0$. Then $\mathfrak{a}$ has an invariant subspace.*

Proof. The theorem covers the case $|\lambda| < 1$. If $|\lambda| > 1$ then $T^*K^* = \lambda^{-1}K^*T^*$ and $K^* \in \mathcal{C}_c(T^*)$. The theorem then shows that $\{T^*\}'_c$ has an invariant subspace. Note that $\mathfrak{a} = \{B\colon B^* \in \{T^*\}'_c\}$. Hence $\mathfrak{a}^*$ has an invariant subspace, and because of the reflexivity of $\mathfrak{X}$ so does $\mathfrak{a}$.

Question. Is it possible to show the existence of an invariant subspace for $\{T\}'_c$ under the weaker assumption that the closure (in some appropriate topology) of $\{T\}'_{sc}$ contains a nonzero compact operator? A reasonable first step might be to show that the weaker condition yields a hyperinvariant subspace for T.

We remark that C. K. Fong **[2]** has recently obtained some related results concerning common invariant subspaces of T and K, under more general conditions than those discussed here.

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