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THE IMAGE OF THE AHLFORS FUNCTION

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ABSTRACT. Let Ω denote a maximal region on the Riemann sphere for bounded holomorphic functions and $p \in \Omega$. We present a class of examples to show that the complement in the unit disk of the image of the Ahlfors function for Ω and p can be a fairly general discrete subset of the unit disk.

1. Introduction. Let Ω be a region on the Riemann sphere that supports nonconstant bounded holomorphic functions and let $p \in \Omega$. Set $\mathfrak{B} = \{f: f \text{ is holomorphic in } \Omega \text{ and } f(\Omega) \subset B\}$, where $B = \{z: |z| < 1\}$. The Ahlfors function for Ω and p is the unique function h in $\mathfrak{B}$ such that

$$h'(p) = \max_{f \in \mathfrak{B}} \operatorname{Re} f'(p).$$

It is elementary to show that $h(p) = 0$. This paper is concerned with the image, $h(\Omega)$, of the Ahlfors function.

First, we survey the known results. Ahlfors [1] showed that $h(\Omega) = B$ for regions Ω of finite connectivity that have no trivial boundary components. More precisely, he proved that h expresses Ω as an n-sheeted branched covering of B, where n is the order of connectivity of Ω. In the general situation Havinson [5] and Fisher [2] demonstrated that $B \setminus h(\Omega)$ has analytic capacity zero; that is, every bounded holomorphic function defined on $h(\Omega)$ may be extended to a bounded holomorphic function on B. It is not difficult to give an example of a region Ω such that $B \setminus h(\Omega) \neq \varnothing$. For example, let K be a closed subset of B which has analytic capacity zero and $\Omega = B \setminus K$. If $0 \in \Omega$, then the Ahlfors function h for Ω and 0 is the identity function, so $h(\Omega) = B \setminus K$. The question of the size of $B \setminus h(\Omega)$ becomes more interesting if it is required that Ω be a maximal region for bounded holomorphic functions in the sense of Rudin [11]. For such a maximal region Ω, Fisher [3] raised the question of whether the Ahlfors function must map Ω onto B. Roding [9] answered this question in the negative by exhibiting a maximal region Ω and a point $p \in \Omega$ such that the Ahlfors function for Ω and p omitted two values in B. We shall extend Roding's result by showing that an Ahlfors function for a maximal region can actually omit a fairly general discrete set of values in B.

2. The example. Suppose K is a discrete subset of B such that $K \cap \mathbf{R} = \varnothing$ and $\overline{K} = K$, where $\overline{K}$ denotes the reflection of the set K in the real axis. Set $\Delta = B \setminus K$ and $\Delta^+ = \Delta \cap H$, where $H = \{z: \operatorname{Im}(z) > 0\}$. Observe that the open interval

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$(-1, 1)$ is a free boundary arc of Δ^+. Let $f\colon \overline{H} \to \Delta^+$ be an analytic universal covering projection of the lower half-plane $\overline{H}$ onto Δ^+; of course, f is not uniquely determined. The function f could be replaced by $f \circ T$, where T is any conformal automorphism of $\overline{H}$. Let Γ be the associated group of cover transformations; that is, Γ is the group of all Möbius transformations T which map $\overline{H}$ onto itself and satisfy $f \circ T = f$. Later, we shall need the fact that Γ possesses a set of generators each of which is parabolic. Let us establish this result now. Fix some point $b \in \Delta^+$. Because $K \cap H$ is discrete, the fundamental group $\pi_1(\Delta^+, b)$ is generated by countably many Jordan loops γ_i with the property that each γ_i contains exactly one point of $K \cap H$ in its interior. Recall that Γ and $\pi_1(\Delta^+, b)$ are isomorphic as follows. Fix a point $\tilde{b} \in \overline{H}$ with $f(\tilde{b}) = b$. For $\gamma \in \pi_1(\Delta^+, b)$ let $\tilde{\gamma}$ be the unique lift of γ having initial point $\tilde{b}$. The terminal point of $\tilde{\gamma}$ also lies over b and there is a unique $T_\gamma \in \Gamma$ which sends $\tilde{b}$ to the terminal point of $\tilde{\gamma}$. The mapping $\gamma \to T_\gamma$ is an isomorphism. Since each γ_i is retractable to an isolated boundary point, it follows that T_{γ_i} is parabolic [7]. Thus, Γ is generated by the T_{γ_i}, each of which is parabolic.

We use f to construct another covering via the Schwarz reflection principle. Because $(-1, 1)$ is a free boundary arc of Δ^+, there is an open set σ contained in the extended real line $R \cup \{\infty\}$ such that f extends continuously to $\overline{H} \cup \sigma$ and f maps each component of σ homeomorphically onto $(-1, 1)$. Without loss of generality, we may assume that $\infty \in \sigma$ and that $f(\infty) = 0$. Let σ_∞ be the component of σ that contains ∞. Note that σ is invariant under the group Γ; in fact, $\sigma = \bigcup T(\sigma_\infty)$, where the union is taken over all $T \in \Gamma$. We extend f to a holomorphic function on $\Omega = \overline{H} \cup \sigma \cup H$ by means of the Schwarz reflection principle: $f(\bar{z}) = \overline{f(z)}$. We continue to denote the extended holomorphic function by f. It is elementary to verify that $f\colon \Omega \to \Delta$ is an analytic covering, that $f'(\infty) > 0$ and that the group of cover transformations associated with this covering is exactly Γ.

Proposition 1. *Ω is a maximal region for bounded holomorphic functions.*

Proof. Suppose that Ω were not maximal. Let Ω^* be the unique maximal region on the Riemann sphere which contains Ω and has the property that every bounded holomorphic function on Ω extends to a bounded holomorphic function on Ω^*. Obviously, any point of $\Omega^* \setminus \Omega$ lies on the real axis. Now, the covering projection f itself extends to a holomorphic function $f^*\colon \Omega^* \to B$ and $f^*(\bar{z}) = \overline{f^*(z)}$. Select $a \in \Omega^* \setminus \Omega$. Because f^* is symmetric about the real axis, $f^*(a) = b \in (-1, 1)$. Select $r > 0$ so small that the closed disk $D = \{z\colon |z - b| \leqslant r\}$ is evenly covered by f. This means that $f^{-1}(D) = \bigcup D_i$, where D_i is a compact subset of Ω, $D_i \cap D_j = \varnothing$ if $i \neq j$ and $f|D_i$ is a homeomorphism of D_i onto D. We claim that the sets D_i cannot cluster at a. If they did, then we could find a sequence $(z_n)_{n=1}^{\infty}$ in Ω with $z_n \in \partial D_{i(n)}$ and $z_n \to a$. This would yield $f^*(z_n) \to b$, a contradiction, since $r = |f(z_n) - b|$ for all n. Hence, we can find an open neighborhood V of a in Ω^* such that $V \cap f^{-1}(D) = \varnothing$ and $f^*(V) \subset D$. Then $V \cap \Omega = \varnothing$ since $z \in V \cap \Omega$ implies $f^*(z) \in D$ which would give $z \in f^{-1}(D)$, a contradiction. Consequently, $V \subset \Omega^* \setminus \Omega \subset \mathbf{R}$, which violates the fact that V is open. Hence, Ω is maximal.

LEMMA. *Let $ax^2 + bx + c$ be a real quadratic with distinct real roots r and s, where $r < s$. Suppose E_0 is a measurable set whose closure is contained in (r, s). Then*

$$\int_E \frac{dx}{ax^2 + bx + c} = 0,$$

where $E = \bigcup_{n\in\mathbf{Z}} E_n$ and $E_n = E_0 + n(s - r) = \{x + n(s - r): x \in E_0\}$.

PROOF. Without loss of generality, we may assume that $a = 1$. Our first step is to show that we may specialize to the situation in which $r = -1$ and $s = 1$. Let

$$y = y(x) = \frac{2x - r - s}{s - r}, \qquad x = x(y) = \tfrac{1}{2}[(s - r)y + (s + r)].$$

Note that $y(r) = -1$ and $y(s) = 1$. Then

$$\int_{E_n} \frac{dx}{x^2 + bx + c} = \int_{E_n} \frac{dx}{(x - r)(x - s)} = \frac{2}{s - r}\int_{F_n} \frac{dy}{y^2 - 1},$$

where $F_0 = \{y(x): x \in E_0\}$ and $F_n = F_0 + 2n$. The closure of F_0 is contained in $(-1, 1)$. Thus, it suffices to show that

$$0 = \int_F \frac{dy}{y^2 - 1} = \sum_{-\infty}^{\infty} \int_{F_n} \frac{dy}{y^2 - 1}, \tag{1}$$

where $F = \bigcup_{n\in\mathbf{Z}} F_n$.

Next, we establish (1) in the special case $F_0 = (a, b)$, where $-1 < a < b < 1$. By direct calculation we obtain

$$\int_{F_0} \frac{dy}{y^2 - 1} = -\frac{1}{2}\log\left(\frac{1 + b}{1 - b}\cdot\frac{1 - a}{1 + a}\right),$$

$$\int_{F_n} \frac{dy}{y^2 - 1} = \log\left(\frac{b + 2n - 1}{b + 2n + 1}\cdot\frac{a + 2n + 1}{a + 2n - 1}\right), \qquad n \neq 0.$$

Then

$$\sum_{n=-N}^{N} \int_{F_n} \frac{dy}{y^2 - 1} = \frac{1}{2}\log\left(\frac{a + 2N + 1}{a - 2N - 1}\cdot\frac{b - 2N - 1}{b + 2N + 1}\right),$$

which implies that (1) is valid in this special case.

Clearly, the preceding work implies that (1) also holds in case F_0 is a finite union of open intervals. Now, consider any measurable set F_0 whose closure is contained in $(-1, 1)$. Let $\varepsilon > 0$ be given. Select $u \in (0, 1)$ so that the closure of F_0 is contained in $(-u, u)$. Next, determine I_0, a finite union of open intervals contained in $(-u, u)$, such that

$$m(F_0 \triangle I_0) < \eta = \varepsilon / \sum_{-\infty}^{\infty} \frac{1}{|(2|n| - u)^2 - 1|},$$

where $F_0 \triangle I_0$ denotes the symmetric difference of the sets F_0 and I_0 and m denotes Lebesgue measure [10, p. 62]. Set $I_n = I_0 + 2n$. Then

$$\left|\sum_{-\infty}^{\infty} \int_{F_n} \frac{dy}{y^2 - 1}\right| = \left|\sum\left(\int_{F_n} \frac{dy}{y^2 - 1} - \int_{I_n} \frac{dy}{y^2 - 1}\right)\right| \leqslant \sum_{-\infty}^{\infty} \int_{F_n \triangle I_n} \frac{dy}{|y^2 - 1|}.$$

Elementary estimates show that

$$\int_{F_n \triangle I_n} \frac{dy}{|y^2 - 1|} < \frac{\eta}{|(2|n| - u)^2 - 1|}.$$

Thus,

$$\left| \sum_{-\infty}^{\infty} \int_{F_n} \frac{dy}{y^2 - 1} \right| \leqslant \varepsilon,$$

so the proof is complete.

PROPOSITION 2. *If h is the Ahlfors function for Ω and ∞, then $h \circ T = h$ for all $T \in \Gamma$.*

PROOF. In order to prove that $h \circ T = h$, it suffices to show that $(h \circ T)'(\infty) = h'(\infty)$ since the Ahlfors function is unique. Let $E = \mathbf{R} \setminus \sigma$. Then E is a compact subset of $\mathbf{R}$ and $T(E) = E$ for all $T \in \Gamma$. A result of Pommerenke [**8**] implies that

$$h(z) = \frac{\exp(g(z)) - 1}{\exp(g(z)) + 1},$$

where $g(z) = \frac{1}{2} \int_E d\zeta/(z - \zeta)$. Note that $g(\infty) = 0$ and $g'(\infty) = (1/2)m(E)$, where $m(E)$ denotes the Lebesgue measure of E as a subset of $\mathbf{R}$. Consequently, $h'(\infty) = \frac{1}{4} m(E)$ and

$$(h \circ T)'(\infty) = \frac{2 \exp(g \circ T(\infty))}{[\exp(g \circ T(\infty)) + 1]^2} (g \circ T)'(\infty).$$

Clearly, it is enough to demonstrate that $g \circ T(\infty) = 0$ and $(g \circ T)'(\infty) = (1/2)m(E)$.

First, we demonstrate that $(g \circ T)'(\infty) = g'(\infty)$. Suppose $T \in \Gamma$ and T is not the identity, say $T(z) = (az + b)/(cz + d)$, where $a, b, c, d \in \mathbf{R}$ and $ad - bc = 1$. Note that $c \neq 0$; otherwise, T fixes the point ∞ which implies that T is the identity because the group Γ is fixed point free on Ω. Clearly, $T(\infty) = a/c$. Now, we calculate $(g \circ T)'(\infty) = g'(a/c)T'(\infty)$. From the definition of g, we obtain

$$g'\left(\frac{a}{c}\right) = -\frac{1}{2} \int_E \frac{d\zeta}{(a/c - \zeta)^2}.$$

We make the change of variable $\omega = T^{-1}(\zeta) = (d\zeta - b)/(-c\zeta + a)$ in this integral and obtain

$$g'\left(\frac{a}{c}\right) = -\frac{1}{2} \int_E c^2 \, d\omega = -\frac{c^2}{2} m(E).$$

Here we have used the facts that $T(E) = E$ and T preserves direction on E because $T(H) = H$. From $T'(\infty) = -1/c^2$, we obtain $(g \circ T)'(\infty) = \frac{1}{2} m(E) = g'(\infty)$.

Second, we demonstrate that $g \circ T(\infty) = 0$ for any parabolic element T of the group Γ. We assume that T has the same form as earlier. The fact that T is parabolic implies that $a + d = \pm 2$; without loss of generality we may assume that

$a + d = 2$. Then the unique fixed point of T is $(a-1)/c \in E$. We want to show that

$$g \circ T(\infty) = g\left(\frac{a}{c}\right) = \frac{1}{2}\int_E \frac{d\zeta}{a/c - \zeta} = 0.$$

We shall make a change of variable in this integral which has the effect of converting T into a translation. Define

$$R(z) = \frac{1}{(a-1)/c - z};$$

then $R^{-1}(z) = (a-1)/c - 1/z$. We make the change of variable $\zeta = R^{-1}(\omega)$ and obtain

$$g \circ T(\infty) = \frac{c}{2}\int_F \frac{d\omega}{\omega(\omega + c)},$$

where $F = R(E)$ is invariant under $U(z) = R \circ T \circ R^{-1}(z) = z - c$. Let F_0 be the part of F that is contained in the open interval between $0 = R(\infty)$ and $-c = R(a/c)$; F_0 is a compact subset of this open interval. Also, $F = \bigcup_{n \in \mathbf{Z}} F_n$, where $F_n = F_0 + nc$, since F is invariant under U. By applying the preceding lemma, we may conclude that $g \circ T(\infty) = 0$.

The foregoing results imply that $h \circ T = h$ for any parabolic element T of Γ. But, as we have already observed, Γ is generated by parabolic elements. It now follows that $h \circ T = h$ for all $T \in \Gamma$.

PROPOSITION 3. *If K is a discrete subset of B, then the analytic covering projection f: $\Omega \to \Delta$ is the Ahlfors function for Ω and ∞.*

PROOF. Let h be the Ahlfors function for Ω and ∞. Then $h'(\infty) \geqslant f'(\infty)$. Proposition 2 shows that the function h is invariant under the group Γ. This implies that h induces an analytic function $\tilde{h}$: $\Delta \to B$ such that $h = \tilde{h} \circ f$. In fact, we may simply define $\tilde{h}(w) = h(z)$, where $z \in \Omega$ is any point such that $f(z) = w$. The function $\tilde{h}$ is well defined since $f(z_1) = f(z_2)$ if and only if there exists $T \in \Gamma$ with $z_2 = T(z_1)$. Because the set K is discrete, the Ahlfors function for Δ and the origin is the identity function. Hence, $\tilde{h}'(0) \leqslant 1$, so that $h'(\infty) = \tilde{h}'(0)f'(\infty) \leqslant f'(\infty)$. This gives $f = h$ since the Ahlfors function is unique.

3. Summary and questions. Suppose that K is a discrete subset of B such that $\overline{K} = K$ and $K \cap \mathbf{R} = \varnothing$. Then the analytic covering f: $\Omega \to \Delta = B \setminus K$ that was constructed in the preceding section is the Ahlfors function for Ω and ∞. The region Ω is maximal for bounded holomorphic functions and $f(\Omega) = B \setminus K$. This is an improvement of the example of Roding [9] in which K consisted of two points. Hence, the Ahlfors function for a maximal region can omit a discrete set of values. Recall that Havinson [5] and Fisher [2] have shown that the set of omitted values always has analytic capacity zero. Therefore, it is still an open question whether the Ahlfors function for a maximal region can actually omit an uncountable set of zero analytic capacity. Gamelin [4] has obtained some results on the range of the Ahlfors function.

Other questions naturally suggest themselves. As usual, Ω denotes a maximal region for bounded holomorphic functions and h an Ahlfors function for Ω. We know that h need not map Ω onto B. How does Ω cover $h(\Omega)$? For example, if Ω is infinitely connected, does each point of $h(\Omega)$ have infinitely many preimages? Also, does the function $h: \Omega \to h(\Omega)$ belong to Bl or Bl_1? Here Bl and Bl_1 are the classes introduced by Heins [6] and generalize the notions of inner function and Blaschke product to mappings between Riemann surfaces. In our example, the function f has the property that each point of $f(\Omega)$ has infinitely many preimages and the function f also belongs to the class Bl_1 since it is a covering.

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