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A MONOTONIC PROPERTY FOR THE ZEROS OF ULTRASPHERICAL POLYNOMIALS

ANDREA LAFORGIA[1]

ABSTRACT. It is shown that $\lambda x_{n,k}^{(\lambda)}$ increases as λ increases for $0 < \lambda < 1$, $k = 1, 2, \ldots, [\frac{n}{2}]$, where $x_{n,k}^{(\lambda)}$ is the kth positive zero of ultraspherical polynomial $P_n^{(\lambda)}(x)$.

The aim of this work is to prove the following

THEOREM. *Let $x_{n,k}^{(\lambda)}$, $k = 1, 2, \ldots, [\frac{n}{2}]$, be the zeros of the ultraspherical polynomial $P_n^{(\lambda)}(x)$ in decreasing order on* $(0, 1)$, *where* $0 < \lambda < 1$.

Then for every $\varepsilon > 0$,

$$\lambda x_{n,k}^{(\lambda)} < (\lambda + \varepsilon) x_{n,k}^{(\lambda+\varepsilon)}, \qquad k = 1, 2, \ldots, \left[\tfrac{n}{2}\right].$$

REMARK 1. For our purposes the following form of Sturm comparison theorem will prove useful. This formulation differs from the usual formulation [2, p. 19] in that $f(x) < F(x)$ is hypothesized for the interval $a < x < X_m$, rather than for the larger interval $a < x < x_m$. (See the work [1] for the proof of this formulation of Sturm theorem.)

LEMMA. *Let the functions $y(x)$ and $Y(x)$ be nontrivial solutions of the differential equations*

$$y''(x) + f(x)y(x) = 0; \qquad Y''(x) + F(x)Y(x) = 0$$

and let them have consecutive zeros at $x_1, x_2, \ldots, x_m$ and $X_1, X_2, \ldots, X_m$ respectively on an interval (a, b). Suppose that $f(x)$ and $F(x)$ are continuous, that $f(x) < F(x)$, $a < x < X_m$, and that

$$\lim_{x \to a^+} \left[y'(x)Y(x) - y(x)Y'(x) \right] = 0. \tag{1}$$

Then

$$X_k < x_k, \qquad k = 1, 2, \ldots, m.$$

PROOF OF THE THEOREM. The function

$$u(x) = (1 - x^2)^{\lambda/2 + 1/4} P_n^{(\lambda)}(x)$$

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which has the same zeros on $(-1, 1)$ as $P_n^{(\lambda)}(x)$ satisfies [2, p. 82] the differential equation

$$y''(x) + p_\lambda(x)y(x) = 0$$

where

$$p_\lambda(x) = \frac{(n+\lambda)^2}{1-x^2} + \frac{2 + 4\lambda - 4\lambda^2 + x^2}{4(1-x^2)^2}.$$

The functions $u(x/\lambda)$ and $u(x/(\lambda+\varepsilon))$ that have on the interval $(0, \lambda)$ and $(0, \lambda+\varepsilon)$ consecutive zeros at $\lambda x_{n,1}^{(\lambda)}, \dots, \lambda x_{n,[n/2]}^{(\lambda)}$ and $(\lambda+\varepsilon)x_{n,1}^{(\lambda+\varepsilon)}, \dots, (\lambda+\varepsilon)x_{n,[n/2]}^{(\lambda+\varepsilon)}$ respectively, satisfy the differential equations

$$z''(x) + \psi_\lambda(x)z(x) = 0, \qquad W''(x) + \psi_{\lambda+\varepsilon}(x)w(x) = 0$$

where $\psi_\nu(x) = \nu^{-2}p_\nu(\frac{x}{\nu})$.

It is easy to show that $\psi_\lambda(x)$ decreases as λ increases for $0 < \lambda < 1$ and $0 < x < \lambda$, that is $\psi_\lambda(x) > \psi_{\lambda+\varepsilon}(x)$, $\varepsilon > 0$. The limit condition (1) is easily shown to hold in the present case. Thus the above Lemma is applicable and its conclusion gives the desired result.

REMARK 2. Our result contrasts with $x_{n,k}^{(\lambda)} > x_{n,k}^{(\lambda+\varepsilon)}$ which follows from formula $\partial x_{n,k}^{(\lambda)}/\partial\lambda < 0$, $k = 1, 2, \dots, [\frac{n}{2}]$. Putting this result together with the new result gives

$$1 < \frac{x_{n,k}^{(\lambda)}}{x_{n,k}^{(\lambda+\varepsilon)}} < 1 + \frac{\varepsilon}{\lambda}, \qquad k = 1, 2, \dots, \left[\frac{n}{2}\right].$$

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ISTITUTO DI CALCOLI NUMERICI, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO, 10-10123 TORINO, ITALY