PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 4, December 1981

AN INEQUALITY CONCERNING THREE FUNDAMENTAL DIMENSIONS OF PARACOMPACT σ-SPACES

SHINPEI OKA

ABSTRACT. It is shown that $\operatorname{Ind} X \leqslant \dim X + \operatorname{ind} X$ for any nonempty paracompact σ-space.

In this paper a map is a continuous map and a space is a Hausdorff topological space. The symbols dim, Ind and ind usually denote covering dimension, large inductive dimension and small inductive dimension, respectively. As is well known, $\operatorname{ind} X \leqslant \operatorname{Ind} X$ and $\dim X \leqslant \operatorname{Ind} X$ for any normal space X. A space is called a σ-space if it admits a σ-locally finite net (cf. [7]). The aim of this paper is to prove the following inequality.

THEOREM 1. *Let X be a nonempty paracompact σ-space. Then* $\operatorname{Ind} X \leqslant \dim X + \operatorname{ind} X$.

COROLLARY 2. *Let X be a paracompact σ-space with* $\operatorname{ind} X \leqslant 0$. *Then* $\dim X = \operatorname{Ind} X$.

Our arguments are based on Leĭbo's ideas of constructing a special kind of map onto metric spaces and making use of Pasynkov's factorization theorem (see [3]).

The following lemma is implicit in [3], and is generalized in [5] to the case where $\mathcal{U}$ is σ-discrete and in [6] to the case where $\mathcal{U}$ is σ-locally finite.

LEMMA 3. *Let X be a submetrizable space (i.e. X admits a weaker metric topology), and let $\mathcal{U}$ be a countable collection of cozero sets of X. Then there exist a metric space M and a one-to-one map f from X onto M such that $f(U)$ is an open set of M for every $U \in \mathcal{U}$.*

The following lemma is well known (see [2, Lemma 2.3.16]).

LEMMA 4. *Let X be a space and let C, K be disjoint closed sets of X. Let $\mathcal{U}$ be a countable open cover of X such that, for each $U \in \mathcal{U}$, either $\overline{U} \cap C = \varnothing$ or $\overline{U} \cap K = \varnothing$. Then C and K are separated by a closed set S such that $S \subset \bigcup \{\operatorname{Bd} U\colon U \in \mathcal{U}\}$.*

Let X be a paracompact σ-space. Let $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be collections of subsets of X satisfying the following conditions:

(1) $\mathcal{F}$ consists of closed sets of X, and $\mathcal{V}$ consists of open sets of X;

Received by the editors February 16, 1981.

1980 *Mathematics Subject Classification.* Primary 54F45, 54E18.

Key words and phrases. σ-space, covering dimension, large inductive dimension, small inductive dimension.

0002-9939/81/0000-0575/$01.75

(2) $\mathfrak{F}$ is a net of X;

(3) for each i, $\mathcal{V}_i$ is written as $\mathcal{V}_i = \{V_i(F)\colon F \in \mathfrak{F}_i\}$, and $F \subset V_i(F)$ for each $F \in \mathfrak{F}_i$;

(4) $\mathcal{V}_i$ is discrete in X for each i.

Every paracompact σ-space admits such collections. For each i, put

$$F_i = \bigcup \{F\colon F \in \mathfrak{F}_i\} \quad \text{and} \quad V_i = \bigcup \{V\colon V \in \mathcal{V}_i\}.$$

LEMMA 5. *Let X be a nonempty paracompact σ-space and let $\mathfrak{F}$, $\mathcal{V}$ be collections of subsets of X satisfying* (1)–(4) *above. Let Y be a totally normal space and let $f\colon X \to Y$ be a one-to-one onto map such that, for each i, $f(F_i)$ is a closed set and $f(V_i)$ is an open set of Y. Then* $\operatorname{Ind} X \leq \operatorname{Ind} Y + \operatorname{ind} X$.

PROOF. The proof is by induction on $\operatorname{Ind} Y + \operatorname{ind} X$. Suppose $\operatorname{Ind} Y + \operatorname{ind} X = 0$. Let C and K be disjoint closed sets of X. For each i, put $\mathfrak{F}_i(C) = \{F \in \mathfrak{F}_i\colon F \subset W$ for some clopen set W with $W \cap C = \varnothing\}$. For each $F \in \mathfrak{F}_i(C)$, fix such a W and denote it by $W_i(F, C)$. Since $\operatorname{ind} X = 0$, it follows from (2) that $\bigcup_{i=1}^{\infty}\mathfrak{F}_i(C)$ covers $X - C$. Since $\operatorname{Ind} Y = 0$, it follows from assumption that, for each i, there exists a clopen set O_i of Y such that $f(F_i) \subset O_i \subset f(V_i)$. By (4), $V_i(F) \cap f^{-1}(O_i)$ is a clopen set for each $F \in \mathfrak{F}_i$. Now put

$$H_i(C) = \bigcup \left\{W_i(F, C) \cap V_i(F) \cap f^{-1}(O_i)\colon F \in \mathfrak{F}_i(C)\right\}.$$

Then, by (4) again, $H_i(C)$ is a clopen set of X. Clearly $X - C = \bigcup_{i=1}^{\infty} H_i(C)$. Similarly we can find clopen sets $H_i(K)$, $i = 1, 2, \ldots,$ such that $X - K = \bigcup_{i=1}^{\infty} H_i(K)$. Hence, by Lemma 4, C and K are separated by the empty set and, therefore, $\operatorname{Ind} X = 0$.

Now suppose that the lemma is valid when $\operatorname{Ind} Y + \operatorname{ind} X \leq n - 1$, and consider the case when $\operatorname{Ind} Y + \operatorname{ind} X = n$. Let A and B be disjoint closed sets of X. For each i, put $\mathfrak{F}_i(A) = \{F \in \mathfrak{F}_i\colon F \subset W$ for some open set W with $\overline{W} \cap A = \varnothing$ and $\operatorname{ind} \operatorname{Bd} W \leq \operatorname{ind} X - 1\}$. For each $F \in \mathfrak{F}_i(A)$, fix such a W and denote it by $W_i(F, A)$. By (2), $\bigcup_{i=1}^{\infty}\mathfrak{F}_i(A)$ covers $X - A$. Note that for any $X' \subset X$, $f|X'\colon X' \to f(X')$ satisfies the assumption of Lemma 5 with respect to $\mathfrak{F}|X'$ and $\mathcal{V}|X'$. Since

$$\operatorname{Ind} f(\operatorname{Bd} W_i(F, A)) + \operatorname{ind} \operatorname{Bd} W_i(F, A) \leq \operatorname{Ind} Y + (\operatorname{ind} X - 1) = n - 1,$$

it follows from the induction hypothesis that $\operatorname{Ind} \operatorname{Bd} W_i(F, A) \leq n - 1$. On the other hand, by assumption, we can find an open set O_i of Y such that $f(F_i) \subset O_i \subset \overline{O}_i \subset f(V_i)$ and $\operatorname{Ind} \operatorname{Bd} O_i \leq \operatorname{Ind} Y - 1$. Since $\operatorname{Ind} \operatorname{Bd} O_i + \operatorname{ind} f^{-1}(\operatorname{Bd} O_i) \leq \operatorname{Ind} Y - 1 + \operatorname{ind} X = n - 1$, it follows from the induction hypothesis that $\operatorname{Ind} \operatorname{Bd} f^{-1}(O_i) \leq \operatorname{Ind} f^{-1}(\operatorname{Bd} O_i) \leq n - 1$. By (4), $\operatorname{Bd}(V_i(F) \cap f^{-1}(O_i)) \subset \operatorname{Bd} f^{-1}(O_i)$. Now put

$$H_i(A) = \bigcup \left\{W_i(F, A) \cap V_i(F) \cap f^{-1}(O_i)\colon F \in \mathfrak{F}_i(A)\right\}.$$

Since

$$\begin{aligned}&\operatorname{Ind} \operatorname{Bd}\big(W_i(F, A) \cap V_i(F) \cap f^{-1}(O_i)\big)\\ &\qquad \leq \max\left\{\operatorname{Ind} \operatorname{Bd} W_i(F, A), \operatorname{Ind} \operatorname{Bd}\big(V_i(F) \cap f^{-1}(O_i)\big)\right\} \leq n - 1,\end{aligned}$$

it follows from (4) that Ind Bd $H_i(A) \leqslant n - 1$. Clearly $X - A = \bigcup_{i=1}^{\infty} H_i(A)$ and $A \cap \operatorname{Cl} H_i(A) = \varnothing$ for each i. Similarly there exist open sets $H_i(B)$, $i = 1, 2, \ldots$, such that Ind Bd $H_i(B) \leqslant n - 1$, $X - B = \bigcup_{i=1}^{\infty} H_i(B)$ and $B \cap \operatorname{Cl} H_i(B) = \varnothing$ for each i. Then, applying Lemma 4, we have a closed set S separating A and B such that $S \subset (\bigcup_{i=1}^{\infty} \operatorname{Bd} H_i(A)) \cup (\bigcup_{i=1}^{\infty} \operatorname{Bd} H_i(B))$. By the countable sum theorem for Ind, we have Ind $S \leqslant n - 1$. Thus Ind $X \leqslant n$, which completes the proof of Lemma 5.

Proof of Theorem 1. Let X be a nonempty paracompact σ-space, and let $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be collections of subsets of X satisfying (1)–(4) above. By [**1**, Lemma 8.2] and [**7**, Theorem 4.6], X is submetrizable. Thus Lemma 3 applies to yield a metric space L and a one-to-one onto map $g: X \to L$ such that, for each i, $g(F_i)$ is a closed set and $g(V_i)$ is an open set of L. By Pasynkov's factorization theorem [**8**, Theorem 29], there exist a metric space M and onto maps $f: X \to M$ and $h: M \to L$ such that $g = h \circ f$ and $\dim M \leqslant \dim X$. Clearly f is one-to-one, $f(F_i)$ is a closed set of M for each i, and $f(V_i)$ is an open set of M for each i. Hence, by Lemma 5,

$$\operatorname{Ind} X \leqslant \operatorname{Ind} M + \operatorname{ind} X = \dim M + \operatorname{ind} X \leqslant \dim X + \operatorname{ind} X.$$

This completes the proof of Theorem 1.

In the proof of Lemma 5, the following theorem is essentially proved.

Theorem 6. *A paracompact σ-space with* ind $X \leqslant 0$ *admits a special family in the sense of Leĭbo* [**4**].

Remark. The inequality in Theorem 1 does not hold even if X is compact. In fact, Filippov [**9**] obtained compact spaces R_i, $i = 1, 2, 3, \ldots$, such that $\dim R_i = 1$, ind $R_i = i$ and Ind $R_i = 2i - 1$. Corollary 2 is also restricted in generalization by Nagami's example [**10**] of a normal space Z with ind $Z = 0$, dim $Z = 1$, Ind $Z = 2$.

References

1. C. J. R. Borges, *On stratifiable spaces*, Facific J. Math. **17** (1966), 1–16.
2. R. Engelking, *Dimension theory*, North-Holland, Amsterdam, 1978.
3. I. M. Leĭbo, *On the equality of dimensions for closed images of metric spaces*, Soviet Math. Dokl. **15** (1974), 835–839.
4. ______, *On closed images of metric spaces*, Soviet Math. Dokl. **16** (1975), 1292–1295.
5. S. Oka, *Dimension of finite unions of metric spaces*, Math. Japon. **24** (1979), 351–362.
6. ______, *A generalization of free L-spaces*, Tsukuba J. Math. (to appear).
7. A. Okuyama, *Some generalizations of metric spaces, their metrization theorems and product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **9** (1967), 236–254.
8. B. A. Pasynkov, *On the spectral decomposition of topological spaces*, Amer. Math. Soc. Transl. (2) **73** (1966), 83–134.
9. V. V. Filippov, *On bicompacta with noncoinciding inductive dimensions*, Soviet Math. Dokl. **11** (1970), 635–638.
10. K. Nagami, *A normal space Z with* ind Z = 0, dim Z = 1, Ind Z = 2, J. Math. Soc. Japan **18** (1966), 158–165.

Department of Mathematics, Kanagawa University, Rokkakubashi, Kanagawa-ku, Yokohama, 221 Japan