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BANACH BUNDLES AND A THEOREM OF J. M. G. FELL

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ABSTRACT. We apply a theorem of J. M. G. Fell to construct the regular representation of a topological groupoid.

1. Introduction. In attempting to place Fell's work on induced representations of locally compact groups [2], [3] in a more categorical framework, one is led naturally to consider actions of topological groupoids on Banach bundles. Some preliminary results in this programme have been announced in [8] and full details will appear elsewhere. However, a natural question which arises here is that of the existence of a representation analogous to the regular representation of a locally compact group. The purpose of this note is to construct such a representation. Moreover, our methods provide what appears to be the correct model of the L^p Banach spaces in the category of Banach bundles, but this point of view will be justified elsewhere.

2. Notation and the construction.

2.1. The notation and terminology is that of [7], and throughout G will denote a locally compact Hausdorff topological groupoid over X. We let $\{m, \mu, \mu_x\}$ denote a fixed Haar measure on G. As usual $\mathrm{St}(x)$ denotes the set $\pi^{-1}(x)$ and $\mathrm{Ct}(x)$ its dual, i.e. the set $\pi'^{-1}(x)$ for each $x \in X$. Each μ_x is a Baire measure defined on the (locally compact Hausdorff) space $\mathrm{Ct}(x)$ which on occasions will be thought of as a Baire measure on G with support contained in $\mathrm{Ct}(x)$. It will ease notation sometimes to denote $\int_{\mathrm{Ct}(x)} d\mu_x$ simply by $\int d\mu_x$, using the suffix x on μ_x to identify the domain of integration. Finally, for any locally compact space Y we denote by $L(Y)$ the space of continuous complex valued functions on Y having compact support.

It will be necessary to make a standing hypothesis concerning $\{\mu_x\}$ as follows.

(I) $\{\mu_x\}$ is *continuous* in the sense that the function $\int f\, d\mu_x$ mapping X into $\mathbf{C}$ is continuous for each $f \in L(G)$, where $\mathbf{C}$ denotes the set of complex numbers. In other words, the function $x \mapsto \mu_x$ is continuous when the cone $M_+(G)$ of all positive measures on G is endowed with the vague topology of Bourbaki.

As shown by J. Westman, see [5], it follows from (I) that π' is an open mapping. Notice that (I) is satisfied in the case that $G = X \times F$, where F is a topological group acting on the right of X. And also in the case that the restriction of π to $\mathrm{St}(x)$ is an open map of $\mathrm{St}(x)$ into X for each $x \in X$, see [6].

2.2. *The construction.* For each $x \in X$ let H_x be the Hilbert space $\{f\colon \mathrm{Ct}(x) \to \mathbf{C};\ \int |f|^2\, d\mu_x < \infty\}$ equipped with the usual inner product $\langle f, g\rangle_x = \int f\bar{g}\, d\mu_x$ and

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associated norm, $\| \|_x$. Let $H = \bigcup_{x \in X} H_x$ and let $p: H \to X$ be the projection defined by $p^{-1}(x) = H_x$. Given $\phi \in L(G)$ there is an induced cross sectional function $\tilde{\phi}: X \to H$ defined by $\tilde{\phi}(x) = \phi|_{\mathrm{Ct}(x)}$. Then $\tilde{\phi}(x) \in L(\mathrm{Ct}(x))$ and $\tilde{\phi}_1 = \tilde{\phi}_2$ if, and only if, $\phi_1 = \phi_2$. Let $\Gamma = \{\tilde{\phi}; \phi \in L(G)\}$. Clearly, Γ is a complex vector space.

2.3. THEOREM. (1) $\|\tilde{\phi}(x)\|_x$ *is a continuous function on* X *for each* $\tilde{\phi} \in \Gamma$.

(2) *For each* $x \in X$, *the set* $\{\tilde{\phi}(x); \tilde{\phi} \in \Gamma\}$ *is dense in* H_x.

(3) $p: H \to X$ *is a Hilbert bundle as defined in* [2] *and each* $\tilde{\phi} \in \Gamma$ *is a section, i.e. is continuous.*

PROOF. (1) $\|\tilde{\phi}(x)\|_x^2 = \int |\phi|^2 \, d\mu_x$ and so the conclusion follows from the hypothesis (I).

(2) Let C be a compact G_δ in $\mathrm{Ct}(x)$. Then C is a compact set in G but not necessarily a G_δ. By the results of [1] there is a compact G_δ D in G with $C \subseteq D$ and $\mu_x(C) = \mu_x(D)$. Hence there is a monotone decreasing sequence of functions $\phi_n \in L(G)$ such that $\phi_n \to \chi_D$ pointwise, where χ_D denotes the characteristic function of D. Hence $\phi_n \to \chi_C$ μ_x-almost everywhere and it follows from the Lebesgue dominated convergence theorem that $\tilde{\phi}_n(x)$ converges in $\| \|_x$ to χ_C. This is enough, for the result now follows by well-known facts concerning the approximation of L^p functions by continuous ones with compact support.

(3) This follows from (1), (2) and Fell's theorem [2, Proposition 1.6].

Henceforth, H will be endowed with the topology of [2, Proposition 1.6]. Let $\| \|$ be the function on H induced by the family $\{\| \|_x; x \in X\}$ of norms. The previous theorem shows that $\| \|$ is continuous.

2.4. REMARK. Precisely the same result holds true with H_x replaced by $L^p(\mathrm{Ct}(x), \mu_x)$, $1 \leqslant p < \infty$, and so H so constructed provides a natural model of the classical Banach spaces L^p in the category of Banach bundles.

We continue the construction by defining an action of G on H, in other words a map $(\cdot)$: $G \times_X H \to H$ subject to the usual axioms, see [6], where $G \times_X H$ denotes the fibred product $\{(\alpha, h) \in G \times H; \pi(\alpha) = p(h)\}$. Given $\alpha \in G$ suppose $\alpha \in G(x, y)$ (i.e. $\pi(\alpha) = x$, $\pi'(\alpha) = y$) and let $f \in H_x$. Define $(\cdot)$ by

$$(\alpha \cdot f)(\beta) = f(\alpha^{-1}\beta)$$

for all $\beta \in \mathrm{Ct}(y)$. Then $\alpha \cdot f$: $\mathrm{Ct}(y) \to \mathbf{C}$. It is immediate that $\alpha \cdot f$ is Baire measurable and by the invariance property of $\{\mu_x\}$, see [7], we have $\|\alpha \cdot f\|^2 = \int |f(\alpha^{-1}\beta)|^2 \, d\mu_y(\beta) = \int |f(\gamma)|^2 \, d\mu_x(\gamma) = \|f\|^2$. Consequently, $\alpha \cdot f \in H_y$. Let ϕ_α: $H_x \to H_y$ be the map defined by $\phi_\alpha(f) = \alpha \cdot f$. One sees easily that ϕ_α is linear, invertible and inner product preserving for each $\alpha \in G$. Moreover, we have $\phi_{\beta\alpha} = \phi_\beta \phi_\alpha$ whenever $\beta\alpha$ is defined in G and that ϕ_α is the identity operator whenever α is an identity in G. Thus $(\cdot)$ will be a unitary representation, called the *regular representation determined by* $\{m, \mu, \mu_x\}$, as soon as we have shown that $(\cdot)$ is continuous.

2.5. REMARK. Unlike the group case, the effect on the regular representation of changing the Haar measure is complicated and will not be discussed here. This is due to the much more complicated structure of the invariant measures μ_x in the case of a groupoid. Indeed, without the separability conditions imposed in [7] the problem of classifying all Haar measures on G seems intractible. In practice, however, one often has a natural choice of $\{\mu_x\}$ which may be used in constructing H. This is the case, for example, when $G = X \times F$, where F is a topological group acting on X. A Haar measure can be constructed in this case, see [7, p. 35], in which μ_x is (essentially) a copy of a fixed right Haar measure ν on F for each $x \in X$. If one changes ν to ν' and c is the positive scalar such that $\nu = c\nu'$, then $T: H \to H'$ defined fibrewise by $T(f) = \sqrt{c}f$ is a unitary equivalence of representations as defined in [8].

3. Continuity of the regular representation. Recall first that if $p: H \to X$ is any Banach bundle over a locally compact space X and s is any element of H, then there is a section σ of p passing through s, that is, $\sigma(p(s)) = s$. A proof of this remarkable result, due to Douady and dal Soglio-Hérault can be found in the appendix of [3]. It has the consequence, see [4], that if $s_0 \in H$ and σ is a fixed section passing through s_0, then neighbourhoods of the type $U(\sigma|_V, \varepsilon) = \{s \in H; p(s) \in V, \|s - \sigma(p(s))\| < \varepsilon\}$, where $\varepsilon > 0$ and V is a neighbourhood of $p(s_0)$, form a neighbourhood base at s_0. Such a neighbourhood $U(\sigma|_V, \varepsilon)$ might reasonably be called a *tube* over V of radius ε and centre σ.

We have the following elementary result.

3.1. PROPOSITION. *Let $p: H \to X$ be a Banach bundle over X and suppose Γ is a vector space of sections with the property that $\{\gamma(x); \gamma \in \Gamma\}$ is dense in H_x for each x in X. Given any neighbourhood U of $s \in H$, there is a section $\gamma \in \Gamma$, a neighbourhood V of $p(s)$ and a positive number ε such that $s \in U(\gamma|_V, \varepsilon) \subseteq U$.*

The next result is a useful generalisation of the well-known fact that strong continuity of a unitary representation Λ, of a group G, is equivalent to joint continuity of the function $\Lambda_g(x)$ mapping $G \times H$ into H. For instance, checking continuity when forming–say–tensor products or direct sums of representations is made comparatively easy.

3.2. THEOREM. *Suppose G acts on a Banach bundle $p: H \to X$ and each ϕ_α is a linear isometry, where ϕ_α is defined by $\phi_\alpha(f) = \alpha \cdot f$. Suppose also that there is a vector space Γ of sections such that*

(1) *$\{\gamma(x); \gamma \in \Gamma\}$ is dense in H_x for each $x \in X$,*

(2) *for each $\gamma \in \Gamma$ the function $G \to H$ defined by $\alpha \mapsto \alpha \cdot \gamma(\pi(\alpha))$ is continuous.*

Then the action $(\cdot): G \times_X H \to H$ is continuous.

PROOF. Let $\alpha_0 \in G(x_0, y_0)$, $f_0 \in H_{x_0}$ and $s_0 = \alpha_0 \cdot f_0$. Starting with an arbitrary neighbourhood U of s_0 in H we can apply Proposition 3.1 and so it suffices to suppose that a section $\sigma \in \Gamma$ passes through s_0 and to consider a neighbourhood $U = U(\sigma|_V, \varepsilon)$ of s_0. Using the fact [5, Proposition 2.4] that π and π' are open mappings find a neighbourhood O of α_0 in G such that $\pi'(O) \subseteq V$ and let $A = \pi(O)$, then A is a neighbourhood of x_0. Using (1) let $\gamma \in \Gamma$ be such that $\|f_0 - \gamma(x_0)\| < \varepsilon/6$ and let $W = U(\gamma|_A, \varepsilon/3)$, then W is a neighbourhood of f_0 in H.

By (2) the function $\|\alpha \cdot \gamma(\pi(\alpha)) - \sigma(\pi'(\alpha))\|$ is continuous in α and at α_0 the value $\|\alpha_0 \cdot \gamma(x_0) - \sigma(y_0)\| = \|\alpha_0 \cdot \gamma(x_0) - \alpha_0 \cdot f_0\| = \|\gamma(x_0) - f_0\| < \varepsilon/6$ since ϕ_{α_0} is an isometry. Hence, there is a neighbourhood of α_0 in G which can be taken to be O, by shrinking O if necessary, such that $\|\alpha \cdot \gamma(\pi(\alpha)) - \sigma(\pi'(\alpha))\| < 2\varepsilon/3$ for all $\alpha \in O$.

Now suppose $\alpha \in O, f \in W$ and $\alpha \cdot f$ is defined. Since ϕ_α is isometric we have

$$\begin{aligned}\|\alpha \cdot f - \sigma(\pi'(\alpha))\| &\leqslant \|\alpha \cdot f - \alpha \cdot \gamma(\pi(\alpha))\| + \|\alpha \cdot \gamma(\pi(\alpha)) - \sigma(\pi'(\alpha))\| \\ &= \|f - \gamma(p(f))\| + \|\alpha \cdot \gamma(\pi(\alpha)) - \sigma(\pi'(\alpha))\| \\ &< \varepsilon/3 + 2\varepsilon/3 = \varepsilon.\end{aligned}$$

Hence, $\alpha \cdot f \in U$ and it follows that $(\cdot)$ is continuous.

Returning to the regular representation of G, we prove

3.3. THEOREM. *The regular representation of G is continuous.*

PROOF. By Theorems 2.3 and 3.2 it suffices to show that for any $\phi \in L(G)$ the function Φ: $G \to H$ is continuous, where $\Phi(\alpha) = \alpha \cdot \tilde{\phi}(\pi(\alpha))$.

Let $\alpha_0 \in G(x_0, y_0)$ and let $s_0 = \alpha_0 \cdot \tilde{\phi}(x_0)$. Applying Proposition 3.1 again it suffices to suppose that a section $\sigma = \tilde{\theta}$ passes through s_0, where $\theta \in L(G)$, and to consider a basic neighbourhood $U = U(\sigma|_V, \varepsilon)$ of s_0. Now

$$\|\Phi(\alpha) - \sigma(\pi'(\alpha))\|^2 = \int_G |\phi(\alpha^{-1}\beta) - \theta(\beta)|^2 \, d\mu_{\pi'(\alpha)}(\beta).$$

Fix a compact neighbourhood O' of α_0 in G and let A and B denote the supports of ϕ and θ respectively. Since G is Hausdorff the set $\mathfrak{D}$ of pairs of composable elements of G is closed in $G \times G$ and so $(O' \times A) \cap \mathfrak{D}$ is compact in $\mathfrak{D}$. Whence the set D which is the union of B and the image of $(O' \times A) \cap \mathfrak{D}$ under the composition function is compact in G. Moreover, it is easy to see that the integrand above vanishes outside D whenever $\alpha \in O'$.

Let C be a compact neighbourhood of D and, using Urysohn's lemma, let f be a continuous function with value 1 on D and which vanishes outside C. By hypothesis (I) there is a positive number M such that

$$\int f \, d\mu_x \leqslant M$$

for all $x \in X$.

Finally, $|\phi(\alpha^{-1}\beta) - \theta(\beta)|^2$ is continuous in α and β and vanishes identically when $\alpha = \alpha_0$. So by compactness of D there is a neighbourhood $O \subseteq O'$ of α_0 and a neighbourhood $W = \pi'^{-1}(\pi'(O)) \cap C$ of $D \cap \mathrm{Ct}(y_0)$ such that $|\phi(\alpha^{-1}\beta) - \theta(\beta)|^2 < \varepsilon^2/2M$ whenever $\alpha \in O$, $\beta \in W$ and $\alpha^{-1}\beta$ is defined, see [**9**, Theorem 1]. Hence, if $\alpha \in O$

$$\begin{aligned}\|\Phi(\alpha) - \sigma(\pi'(\alpha))\|^2 &= \int_D |\phi(\alpha^{-1}\beta) - \theta(\beta)|^2 \, d\mu_{\pi'(\alpha)}(\beta) \\ &\leqslant \frac{\varepsilon^2}{2M} \int_C f \, d\mu_{\pi'(\alpha)} \leqslant \frac{\varepsilon^2}{2M} M < \varepsilon^2.\end{aligned}$$

So $\Phi(\alpha) \in U$ and therefore Φ is continuous as required.

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REFERENCES

1. S. K. Berberian, *Measure and integration*, Macmillan, New York, 1967.

2. J. M. G. Fell, *An extension of Mackey's method to Banach*-algebraic bundles*, Mem. Amer. Math. Soc. No. 90 (1969).

3. ______, *Induced representations and Banach*-algebraic bundles*, Lecture Notes in Math., vol. 582, Springer-Verlag, Berlin and New York, 1977.

4. K. H. Hofmann, *Bundles of Banach spaces, sheaves of Banach spaces and $C(B)$-modules*, Technische Hochschule, Darmstadt, 1974.

5. J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin and New York, 1980.

6. A. K. Seda, *A continuity property of Haar systems of measures*, Ann. Soc. Sci. Bruxelles **89** (1975), 429–433.

7. ______, *Haar measures for groupoids*, Proc. Roy. Irish Acad. Sect. A **76** (1976), 25–36.

8. ______, *Quelques résultats dans la catégorie des groupoids d'opérateurs*, C. R. Acad. Sci. Paris **288** (1979), 21–24.

9. ______, *Banach bundles of continuous functions and an integral representation theorem*, Trans. Amer. Math. Soc. (to appear).

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