PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 84, Number 1, January 1982

ON ANISOTROPIC SOLVABLE LINEAR ALGEBRAIC GROUPS

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ABSTRACT. A connected linear algebraic solvable group G defined over a field k is anisotropic over k if G has no k-subgroup splitting over k. A simple criterion for anisotropic solvable groups is presented when k is a local field.

Let G be a connected linear algebraic solvable group defined over a field k. The group G is said to be *splitting over* k if G has a normal series of k-subgroups such that the factor groups are k-isomorphic either to the additive group G_a or the multiplicative group G_m. We say that G is *anisotropic over* k if G has no k-subgroups splitting over k. In this note, we give a criterion for anisotropic solvable groups in terms of compactness condition when k is a local field. Our main result is the following theorem.

THEOREM M. *Let G be a connected linear algebraic solvable group defined over a local field k. Then the following conditions are equivalent.*

(i) *G is anisotropic over k.*

(ii) *G is nilpotent, and both the maximal torus T of G and the quotient group G/T are anisotropic over k.*

(iii) *The group $G(k)$ of k-rational points of G is compact where $G(k)$ is endowed with the locally compact topology from that of k.*

When G is a torus, the result is well known. The argument of the next lemma is due to Prasad [2].

LEMMA 1. *Let T be a torus defined over a local field k. Then $T(k)$ is compact if and only if T is anisotropic over k.*

PROOF. We know that T is splitting over a finite Galois extension K of k. Clearly, $T(k)$ is a closed subgroup of $T(K)$. From this $T(k)$ is compact if and only if for every $t \in T(k)$ and character χ of T, $\chi(t)$ is of absolute value 1. If $T(k)$ is not compact, then there exists $t \in T(k)$ such that for at least one character χ of T, $\chi(t)$ has absolute value $\neq 1$. This implies that $\sum_{\sigma \in \mathrm{Gal}(K/k)} \sigma_{\chi(t)}$ also has absolute value $\neq 1$. Thus the character $\sum_{\sigma \in \mathrm{Gal}(K/k)} \sigma_{\chi}$ is nontrivial and defined over k. This shows that T is k-isotropic.

For unipotent groups, we need more lemmas.

Received by the editors April 6, 1981.

1980 *Mathematics Subject Classification*. Primary 20G25.

Key words and phrases. Linear algebraic groups, solvable groups, unipotent groups, local fields.

[1]Partially supported by NSF Grant No. 79-00695.

LEMMA 2. *Let k be a local field with characteristic* $\mathrm{ch}(k) = p > 0$ *and A a subset of k^n. If f is an additive k-morphism of G_a^n such that $f(A)$ is relatively compact in k, then up to a k-automorphism of G_a^n, there exists an integer r with $0 \leq r \leq n$ satisfying the following conditions.*

(i) *f is independent of the first r coordinates.*

(ii) *Let* pr *be the projection of G_a^n onto the last $n - r$ coordinates. The projection* $\mathrm{pr}(A)$ *of A is relatively compact in k^{n-r}.*

PROOF. Clearly, we may assume that f is nontrivial. For $1 \leq i \leq n$, we define an additive k-morphism f_i of G_a by $f_i = f \circ \iota_i$ where ι_i is the inclusion map of G_a into the ith component. Since f is additive, for $x = (x_1, \dots, x_n) \in G_a^n$, we have

$$f(x) = f_1(x_1) + \cdots + f_n(x_n).$$

Denote by I the set of indices j with $f_j \neq 0$. After replacing f by $f \circ \alpha$ where α is a k-automorphism of G_a^n, we may assume that the cardinality of I is minimal. Hence it suffices to show that A is relatively compact when $I = \{1, 2, \dots, n\}$. Suppose that the assertion is false. There exists a sequence $\xi_m = (\xi_1(m), \dots, \xi_n(m))$ of elements in A such that the norms $\|\xi_m\|$ $(m = 1, 2, \dots)$ are not bounded. The maps f_i $(i = 1, \dots, n)$ are additive k-morphisms of G_a. Hence we can write

$$f_i(t) = a_{i,0}t + a_{i,1}t^p + \cdots + a_{i,s_i}t^{p^{s_i}},$$

with $a_{i,s_i} \neq 0$ $(i = 1, \dots, n)$. Here we may assume that the number $\sum_{i=1}^n s_i$ has been chosen to be minimal. After replacing (ξ_m) by a subsequence and up to a k-automorphism of G_a^n, there is a positive integer $\nu \leq n$ satisfying the following conditions.

$$\text{(2.1)}\qquad \begin{aligned} &(1)\ \xi_i(m) \to \infty, \qquad 1 \leq i \leq \nu.\\ &(2)\ \text{For } i, j \leq \nu, \text{ the numbers } p^{s_i}\operatorname{ord}_k(\xi_i(m)) - p^{s_j}\operatorname{ord}_k(\xi_j(m))\\ &\qquad \text{are independent of } m.\\ &(3)\ \text{For } i \leq \nu, j > \nu, \text{ the sequence } p^{s_j}\operatorname{ord}_k(\xi_j(m)) - p^{s_i}\operatorname{ord}_k(\xi_i(m))\\ &\qquad \text{tends to } \infty. \end{aligned}$$

Now let $s = \max\{s_i, \dots, s_\nu\}$ and assume, as we may, that $s = s_1$. Since $f(A)$ is relatively compact in k, by (1) of (2.1), the sequence $f(\xi_m)\xi_1(m)^{-p^{s_1}}$ converges to zero, and by (2) and (3) of (2.1) the sequence b_m,

$$b_m = a_{1,s_1} + a_{2,s_2}\left(\xi_2(m)\xi_1(m)^{-p^{s_1-s_2}}\right)^{p^{s_2}} + \cdots + a_{\nu,s_\nu}\left(\xi_\nu(m)\xi_1(m)^{-p^{s_1-s_\nu}}\right)^{p^{s_\nu}},$$

converges to zero. It follows readily from (2) of (2.1) that there exist $\xi_2, \dots, \xi_\nu \in k$ such that

$$a_{1,s_1} + a_{2,s_2}\xi_2^{p^{s_2}} + \cdots + a_{\nu,s_\nu}\xi_\nu^{p^{s_\nu}} = 0. \tag{2.2}$$

Then we have the identity

$$\begin{aligned} &a_{1,s_1}x_1^{p^{s_1}} + \cdots + a_{\nu,s_\nu}x_\nu^{p^{s_\nu}}\\ &\quad = a_{2,s_2}\left(x_2 - \xi_2 x_1^{p^{s_1-s_2}}\right)^{p^{s_2}} + \cdots + a_{\nu,s_\nu}\left(x_\nu - \xi_\nu x_1^{p^{s_1-s_\nu}}\right)^{p^{s_\nu}}. \end{aligned} \tag{2.3}$$

Thus if we set $x_j' = x_j - \xi_j x_1^{p^{s_1 - s_j}}$ $(j = 2, \ldots, \nu)$ and $x_i' = x_i$, $i \notin \{2, \ldots, \nu\}$, it is easy to verify that in the coordinates $(x_1', \ldots, x_n')$

$$\deg(f_1(x_1')) < \deg(f_1(x_1))$$

and

$$\deg(f_i(x_i')) = \deg(f_i(x_i)), \qquad (1 < i \leqslant n),$$

where deg is the degree of a polynomial. Obviously we arrive at a contradiction to our choice of minimality of $\sum_{i=1}^n s_i$. Therefore A has to be relatively compact in k^n and the lemma is proved.

LEMMA 3. *Let k be as in Lemma 2, A a subset of k^n and $f_1, \ldots, f_l$ additive k-morphisms of G_a^n. Suppose that the images $f_i(a)$ are relatively compact in k $(i = 1, \ldots, l)$. Then G_a^n has a decomposition $G_a^n = H \times L$ defined over k such that $H \simeq G_a^r$, $L \simeq G_a^{n-r}$ over k. $H \subset \ker(f_j)$ $(j = 1, \ldots, l)$ and $\mathrm{pr}_L(A)$ is relatively compact in $L(k)$ where pr_L is the projection map of G_a^n into L.*

PROOF. We may assume that A is not relatively compact in k^n. By Lemma 2, G_a^n has a decomposition $G_a^n = M \times N$ defined over k such that $M \simeq G_a^t$, $N \simeq G_a^{n-t}$ over k, $t > 0$, and $M \subset \ker(f_1)$, and the projection $\mathrm{pr}_N(A)$ of A in N is relatively compact in $N(k)$. Now let $A_1 = \mathrm{pr}_M(A)$. Clearly $A_1, f_2|M, \ldots, f_l|M$ satisfy all the conditions in Lemma 3. By induction on l, our assertion is true in M and consequently in G_a^n.

PROPOSITION 4. *Let k be a local field and G a k-subgroup of G_a^n. Then G_a^n has a decomposition $G_a^n = H \times L$ defined over k such that $H \simeq G_a^r$, $L \simeq G_a^{n-r}$ over k, $H \subset G$ and $(G \cap L)(k)$ is compact.*

PROOF. We may assume that $\mathrm{ch}(k) = p > 0$. By [4, p. 102, Proposition], there exist additive k-morphisms $f_1, \ldots, f_l$ such that $G = \bigcap_{i=1}^l \ker(f_i)$. Now the proposition is an immediate consequence of Lemma 3.

THEOREM 5. *Let G be a connected linear algebraic unipotent group defined over a local field k. The following conditions are equivalent.*

(i) *G is anisotropic over k.*

(ii) *There exist no nontrivial additive k-morphisms from G_a into G.*

(iii) *$G(k)$ is compact.*

PROOF. If $\mathrm{ch}(k) = 0$, G is always k-splitting. In this case, all three conditions are equivalent to $G = \{1\}$. Hence we may assume that $\mathrm{ch}(k) = p > 0$ and prove the theorem in several steps.

Clearly, (iii) $\to$ (i) $\to$ (ii). Thus we show (ii) $\to$ (iii). Condition (ii) is equivalent to the condition that the maximal k-splitting subgroup of G is $\{1\}$.

Step 1. G is commutative and $G^p = \{1\}$. We know [3, p. 34, Corollary 2] that G is isomorphic to G_a^m over $k^{p^{-l}}$ for certain nonnegative integers m, l. Hence there is an isomorphism $G \xrightarrow{\tau} G_a^m$ defined over $k^{p^{-l}}$. Let f: $G \to G_a^m$ be the k-morphism given by $f(x) = \tau(x)^{p^l}$ $(x \in G)$. Clearly, $\ker(f) = \{1\}$. Express τ in the form $\tau = \sum_{\sigma=1}^r \omega_\sigma \tau_\sigma$ where τ_σ are defined over k and ω_σ $(\in k^{p^{-l}})$ are linearly independent over k. It is easy to see that for $x, y \in G(k)$ $\tau_\sigma(x+y) = \tau_\sigma(x) + \tau_\sigma(y)$. Since $G(k)$ is Zariski-dense in G, the maps τ_σ are k-morphisms of G into G_a^m. By assumption on τ, the differential $d\tau$ of τ is an isomorphism, it follows readily that $\bigcap_\sigma \ker(d\tau_\sigma) = \{0\}$. Therefore the map g: $G \to G_a^{rm}$ given by $g(x) = (\tau_\sigma(x))$ $(x \in G)$ is a separable k-morphism. Now using f and g, we define ω: $G \to G_a^{(r+1)m}$ by $\omega(x) = (f(x), g(x))$ $(x \in G)$. Clearly, ω defines a k-embedding of G into $G_a^{(r+1)m}$. From Proposition 4, $G(k)$ has to be compact.

Step 2. Suppose that G has a connected normal k-subgroup N with $\{1\} \neq N \neq G$. Let $L = G/N$, and L' its maximal k-splitting subgroup. If $L' \neq L$, let H be the inverse image of L' in G. By induction on dimension, $H(k)$ and $(G/H)(k)$ are compact. Since the image of $G(k)$ in $(G/H)(k)$ is open, it follows that $G(k)/H(k)$ is compact, thus so is $G(k)$.

Step 3. G is commutative and $G^p \neq \{1\}$. Let l be the largest integer with $G^{p^l} \neq \{1\}$ and $N = G^{p^l}$. Let $L = G/N$ and L' the maximal k-splitting subgroup of L. If $L \neq L'$, by Step 2, $G(k)$ is compact. If $L = L'$, the map $x \mapsto x^p$ $(x \in G)$ factors through L. Then G^p, as a homomorphic image of a k-splitting unipotent group, by [3, p. 35, Proposition 6] is k-splitting. However, $G^p \neq \{1\}$ and by condition (ii), this is impossible.

Step 4. G is not commutative. Let $N = [G, G]$, $L = G/N$ and L' the maximal k-splitting subgroup of L. Suppose that $L = L'$. Let H be the last term in the lower central series with $H \not\subset Z(G)$ where $Z(G)$ is the center of G. Then choose any $h \in H(k)$ such that $h \notin Z(G)$ and consider the map $x \mapsto xhx^{-1}h^{-1}$ $(x \in G)$. The image of the map is in $Z(G)$ by our choice of H, hence is a k-morphism of algebraic groups. It factors through L. Therefore $[h, G]$, by [3, Proposition 6] is k-splitting. By (ii), $[h, G]$ is anisotropic over k, thus $[h, G] = \{1\}$. However $h \notin Z(G)$, we have a contradiction. Therefore $L' \neq L$ and by Step 2, $G(k)$ is compact.

Now are ready to prove our main result.

Proof. When $\mathrm{ch}(k) = 0$, all the three conditions are equivalent to that G is an isotropic torus for $R_u(G)$ is always splitting over k. Hence we may assume that $\mathrm{ch}(k) = p > 0$.

(i) $\to$ (ii). By [4, p. 114, Corollary 2], G is nilpotent. Clearly, T is anisotropic over k. Let H be the maximal k-splitting subgroup of G/T and L its preimage in G. Since T is splitting over a finite separable extension K of k, L is splitting over K. This implies that $R_u(L)$ is defined over K. On the other hand, L is defined over k, so $R_u(L)$ is k-closed. Thus $R_u(L)$ is defined over k. As $R_u(L)$ is k-isomorphic to $L/T = H$, $R_u(L)$ is splitting over k. Therefore $R_u(L) = \{1\}$ and so is $H = \{1\}$.

(ii) $\to$ (iii). From Lemma 1 and Theorem 5, $T(k)$ and $(G/T)(k)$ are compact. We know that the image of $G(k)$ in $(G/T)(k)$ is open, hence compact. It follows readily that $G(k)$ is compact because $T(k)$ and $G(k)/T(k)$ are compact.

(iii) $\to$ (i) is obvious.

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