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# TOPOLOGICALLY UNREALIZABLE AUTOMORPHISMS OF FREE GROUPS

JOHN R. STALLINGS[1]

ABSTRACT. Let $\phi: F \to F$ be an automorphism of a finitely generated free group. It has been conjectured (I heard it from Peter Scott) that the fixed subgroup of $\phi$ is always finitely generated. This is known to be so if $\phi$ has finite order [1], or if $\phi$ is realizable by a homeomorphism of a compact 2-manifold with boundary [2]. Here we give examples of automorphisms $\phi$, no power of which is topologically realizable on any 2-manifold; perhaps the simplest is the automorphism of the free group of rank 3, given by $\phi(x) = y$, $\phi(y) = z$, $\phi(z) = xy$.

**1. PV-matrices and automorphisms.** By a PV-matrix is meant an $n \times n$ integer matrix of determinant $\pm 1$, having one eigenvalue, $\lambda_1$, of absolute value greater than 1, and $n - 1$ eigenvalues of absolute value less than 1. The terminology "PV" is used because $\lambda_1$ is a Pisot-Vijayaraghavan number [3].

A PV-automorphism of a free abelian group of rank $n$ is an automorphism whose matrix is a PV-matrix, (PV-ness is independent of the basis). A PV-automorphism of a free (nonabelian) group, is an automorphism whose abelianization is PV.

1.1. If $M$ is an $n \times n$ PV-matrix and $n \geqslant 3$, then, since $|\lambda_1\lambda_2 \ldots \lambda_n| = 1$, no eigenvalue $\lambda_i$ is the inverse of any $\lambda_j$.

1.2. If $M$ is a PV-matrix, then every positive integral power $M^k$ is a PV-matrix, since the eigenvalues of $M^k$ are the $k$th powers of those of $M$.

A simple $3 \times 3$ example of a PV-matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with eigenvalues approximately

$$\lambda_1 = 1.3247, \qquad \lambda_2, \lambda_3 = -0.6624 \pm 0.5623\sqrt{-1}\,.$$

Correspondingly, $\phi(x) = y$, $\phi(y) = z$, $\phi(z) = xy$ describes a PV-automorphism of the free group with basis $\{x, y, z\}$.

**2. Eigenvalues of automorphisms of a 2-manifold.**

2.1. Let $h: T \to T$ be an orientation-preserving homeomorphism of a closed, orientable 2-manifold onto itself. Then the eigenvalues of the homology map

$$h_*: H_1(T) \to H_1(T)$$

occur in inverse pairs; that is, they can be listed $\lambda_1, \lambda_2, \ldots, \lambda_g, \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_g^{-1}$.

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The reason is that $h_*$ preserves an alternating bilinear form, the intersection number, and thus $h_*$ is symplectic, and every symplectic matrix is similar to its inverse. (Cf. Exercise 1, §6.9, p. 377 in [4].)

2.2. Let $h$: $T \to T$ be an orientation-preserving homeomorphism of a compact, orientable, connected 2-manifold, with $\beta \geqslant 1$ boundary components, onto itself, such that $h$ maps each boundary component to itself. Then the eigenvalues of the homology map

$$h_*: H_1(T) \to H_1(T)$$

consist of two sorts: There are $\beta - 1$ eigenvalues $= 1$, and the remaining eigenvalues occur in inverse pairs.

The reason for this is that the subgroup $S$ of $H_1(T)$ generated by the boundary components is a free abelian direct summand of rank $\beta - 1$; and $h_*$ maps $S$ to itself by the identity. On the quotient by $S$, $h_*$ is the homology map on the closed manifold obtained by capping off the boundary components with 2-cells, and to this we apply 2.1.

### 3. PV-automorphisms are not realizable.

THEOREM. *Let* $\phi$: $F \to F$ *be a PV-automorphism of a free group of infinite rank* $n \geqslant 3$. *Then, for every integer* $k \geqslant 1$, $\phi^k$ *is not realizable, as the automorphism on fundamental group, by any homeomorphism* $h$: $T \to T$ *of any* 2-*manifold* $T$.

PROOF. There are two cases, $T$ orientable or not.

*Orientable case*. If $T$ is orientable, then $h^2$ is orientation preserving and permutes the boundary components of $T$; this permutation has some finite order $q$, so that $h^{2q}$ is orientation preserving and maps each boundary component to itself. The homomorphism

$$h_*^{2q}: H_1(T) \to H_1(T)$$

is the abelianization of $\phi^{2kq}$. This is a PV-automorphism by 1.2, and thus none of the eigenvalues of $h_*^{2q}$ is the inverse of any other by 1.1. This contradicts 2.2.

*Nonorientable case*. If $T$ is nonorientable, let $T' \to T$ be its orientable double cover. The homeomorphism $h$: $T \to T$ lifts to a homeomorphism $h'$: $T' \to T'$. There is a transfer homomorphism

$$\tau: H_1(T) \to H_1(T')$$

such that $\tau \circ h_* = h'_* \circ \tau$, and such that the composition

$$H_1(T) \to H_1(T') \to H_1(T)$$

is multiplication by 2.

If we take the coefficient group to be the field of rational numbers–this does not change any argument on eigenvalues–then $\tau$ embeds $H_1(T)$ as a subspace of $H_1(T')$ which is invariant under $h'_*$ and on which $h'_*$ is isomorphic to $h_*$.

Now, as in the orientable case, there is some positive integer $q$ such that $(h')^{2q}$ is orientation preserving and maps each boundary component of $T'$ to itself. The list of eigenvalues of $(h'_*)^{2q}$ includes, by the transfer argument, the eigenvalues of $h_*^{2q}$. The latter are eigenvalues of the abelianization of $\phi^{2kq}$, which, as before, do not include any inverse pairs.

There are $n$ eigenvalues of $h_*^{2q}$, since the rank of $H_1(T)$ is the rank of $F$, which is $n$. A Euler characteristic argument shows that the rank of $H_1(T')$ is $2n - 1$. Therefore, there are not enough additional eigenvalues of $(h'_*)^{2q}$ to make up a set of eigenvalues satisfying 2.2.

**4. Comments.**

4.1. Suppose that $\phi: F \to F$ is a PV-automorphism and that $S \subset F$ is a subgroup of finite index with $\phi(S) = S$. I suspect that $\phi|S$ is not realizable by a surface homeomorphism and that this can be proved by examining eigenvalues. If $S$ is a normal subgroup and $\phi$ induces the identity on $G = F/S$, then $\phi$ determines a ZG-automorphism on the abelianization of $S$. Can this automorphism have a symmetric set of eigenvalues (satisfying 2.2)? This seems to involve the question: What does the fact that $F$ is free imply about the structure of the abelianization of $S$ as a ZG-module?

4.2. Every automorphism $\phi: F \to F$ of a free group of rank $n$, whose abelianization has determinant $+1$, leaves something fixed modulo the $(n + 1)$st term in the lower central series.

PROOF. Define

$$F_1 = F, \qquad F_{k+1} = [F, F_k].$$

Then the quotients of the lower central series $L_k = F_k/F_{k+1}$ form a free Lie algebra over $Z$, and $\phi$ induces automorphisms $\phi_k$: $L_k \to L_k$. Tensor with the complex numbers $\mathbf{C}$. Then $\phi_1$ has eigenvalues $\lambda_1, \dots, \lambda_n$, and corresponding eigenvectors $\xi_1, \dots, \xi_n$ in $L_1 \otimes \mathbf{C}$. However, we need to have $\xi_{n-1}$ and $\xi_n$ linearly independent, so that if $\lambda_{n-1} = \lambda_n$, it may be necessary to change the defining equation for $\xi_n$ from $\phi_1(\xi_n) = \lambda_n\xi_n$ to $\phi_1(\xi_n) = \lambda_n\xi_n + \xi_{n-1}$.

Then the element

$$\eta = [\xi_1, [\dots, \xi_\eta]] \quad \text{in } L_n \otimes \mathbf{C}$$

is nonzero and has the property that

$$\phi_n(\eta) = \lambda_1 \dots \lambda_n \eta = \eta.$$

Thus $\phi_n$ has an eigenvalue 1, and therefore has an *integral* eigenvector $\theta$ corresponding to the eigenvalue 1. Then $\theta$ is represented by $w \in F_n - F_{n+1}$ such that $\phi(w) \equiv w$ modulo $F_{n+1}$.

For example, taking $\phi$ to be the PV-automorphism described at the end of §1, both

$$[x, [y, z]][x, [x, z]][y, [y, x]][z, [z, y]]$$

and

$$[x, [y, z]][x, [x, y]][x, [x, z]][y, [y, z]][z, [z, x]][z, [z, y]]$$

are fixed modulo $F_4$.

Hence, it is at least conceivable that this automorphism leaves something in $F_3$ fixed. My conjecture, which I cannot prove, is that the fixed subgroup of every PV-automorphism is trivial.

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Department of Mathematics, University of California, Berkeley, California 94720